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Approximation Properties of Dicrete Fourier Sums for Some Piecewise Linear Functions

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Let N be a natural number greater than 1. We select N uniformly distributed points $t_k = 2\pi k/N$ $(0 \leqslant k \leqslant N-1)$ on $[0, 2\pi]$. Denote by $L_{n,N}(f) = L_{n,N}(f,x)$ $(1 \leqslant n \leqslant N/2)$ the trigonometric polynomial of order n possessing the least quadratic deviation from f with respect to the system $\{t_k\}_{k=0}^{N-1}$. In other words, the greatest lower bound of the sums $\sum_{k=0}^{N-1} |f(t_k) - T_n(t_k)|^2$ on the set of trigonometric polynomials T_n of order n is attained by $L_{n,N}(f)$. In the present article the problem of function approximation by the polynomials $L_{n,N}(f,x)$ is considered. Using some example functions we show that the polynomials $L_{n,N}(f,x)$ uniformly approximate a piecewise-linear continuous function with a convergence rate $O(1/n)$ with respect to the variables $x \in \mathbb{R}$ and $1 \leqslant n \leqslant N/2$. These polynomials also uniformly approximate the same function with a rate $O(1/n^2)$ outside of some neighborhood of function's „crease" points. Also we show that the polynomials $L_{n,N}(f,x)$ uniformly approximate a piecewise-linear discontinuous function with a rate $O(1/n)$ with respect to the variables x and $1 \leqslant n \leqslant N/2$ outside some neighborhood of discontinuity points. Special attention is paid to approximation of 2π-periodic functions f_1 and f_2 by the polynomials $L_{n,N}(f,x)$, where $f_1(x) = |x|$ and $f_2(x) = \operatorname{sign} x$ for $x \in [-\pi, \pi]$. For the first function f_1 we show that instead of the estimate $|f_1(x) - L_{n,N}(f_1,x)| \leqslant c \ln n/n$ which follows from the well-known Lebesgue inequality for the polynomials $L_{n,N}(f,x)$ we found an exact order estimate $|f_1(x) - L_{n,N}(f_1,x)| \leqslant c/n$ $(x \in \mathbb{R})$ which is uniform relative to $1 \leqslant n \leqslant N/2$. Moreover, we found a local estimate $|f_1(x) - L_{n,N}(f_1,x)| \leqslant c(\varepsilon)/n^2$ $(|x - \pi k| \geqslant \varepsilon)$ which is also uniform relative to $1 \leqslant n \leqslant N/2$. For the second function f_2 we found only a local estimate $|f_2(x) - L_{n,N}(f_2,x)| \leqslant c(\varepsilon)/n$ $(|x - \pi k| \geqslant \varepsilon)$ which is uniform relative to $1 \leqslant n \leqslant N/2$. The proofs of these estimations are based on comparing of approximating properties of discrete and continuous finite Fourier series.

Key words: function approximation, trigonometric polynomials, Fourier series.

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