An Asymptotic Relation for Conformal Radii of Two Nonoverlapping Domains

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We consider a family of continuously varying closed Jordan curves given by a polar equation, such that the interiors of the curves form an increasing or decreasing chain of domains. Such chains can be described by the Löwner – Kufarev differential equation. We deduce an integral representation of a driving function in the equation. Using this representation we obtain an asymptotic formula, which establishes a connection between conformal radii of bounded and unbounded components of the complement of the Jordan curve when the bounded component is close to the unit disk.

Key words: Löwner – Kufarev equation, conformal radius, asymptotic expansion, nonoverlapping domains.

DOI: https://doi.org/10.18500/1816-9791-2018-18-3-274-283

INTRODUCTION

We denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ the open disk of the radius $r > 0$ and centered at the origin, $\mathbb{D} = \mathbb{D}_1$. Let Ω be a simply connected domain which is a proper subset of the complex plane and $w_0 \in \Omega$. According to the Riemann mapping theorem there are a unique number $r > 0$ and a unique function g conformally mapping Ω onto the disk $\mathbb{D}_r$ and such that $g(w_0) = 0$, $g'(w_0) = 1$. This r is called a conformal radius of the domain Ω with respect to the point w_0. Let now Ω be a domain in the extended complex plane with at least two boundary points and $w_0 = \infty \in \Omega$. There are a unique $r > 0$ and a unique function g, which is analytic in Ω except ∞, where it has the expansion $g(w) = w + c_0 + c_1 w^{-1} + ...$, and maps Ω one-to-one onto $\{|z| > \frac{1}{r}\}$. This r is called a conformal radius of the domain Ω with respect to ∞. In both cases we denote the conformal radius of Ω with respect to the point w_0 by $r(\Omega, w_0)$. Note that if f maps conformally the unit disk $\mathbb{D}$ onto $\Omega \subset \mathbb{C}$ and $f(0) = w_0$, $f'(0) > 0$, then $r(\Omega, w_0) = f'(0)$.

Let Ω_1, Ω_2 be disjoint simply connected domains in the extended complex plane, $0 \in \Omega_1$, $\infty \in \Omega_2$. Then $r(\Omega_1, 0) r(\Omega_2, \infty) \leqslant 1$, where the equality sign holds if and only if Ω_1 and Ω_2 are the bounded and unbounded components of the complement of a circle with the center at the origin. This results in a corollary of the theorem about nonoverlapping domains obtained by N. A. Lebedev using the area principle [1]. We will consider the case when Ω_1 and Ω_2 are the bounded and unbounded components of a closed Jordan curve, respectively. Let $f : \mathbb{D} \to \Omega_1$ and $F : \{|z| > 1\} \to \Omega_2$ be conformal maps. The composition $F^{-1} \circ f$ determines a homeomorphism of the unit circle which is called a conformal welding. We refer the reader to the works [2–7].

In the article, we use the Löwner – Kufarev parametric method to establish an asymptotic relation for conformal radii of two nonoverlapping domains. The Löwner equation is a differential equation describing a continuously increasing sequence of simply connected domains of a special type, i.e. the so called slit domains [8]. Kufarev [9] and Pommerenke [10] generalized the Löwner equation to a wider class of domains.

Given a chain of simply connected domains $\Omega(t)$, $t \in [0, T)$, such that $0 \in \Omega(t_1) \subset \subset \Omega(t_2)$, $0 \leqslant t_1 < t_2 < T$, the function $f(z,t) = e^t z + \ldots$, conformally mapping $\mathbb{D}$ onto $\Omega(t)$ for each fixed $t \in [0,T)$, a.e. satisfies the (Löwner – Kufarev) equation [9, 11]

$$\frac{\partial f(z,t)}{\partial t} = z\frac{\partial f(z,t)}{\partial z} p(z,t), \quad z \in \mathbb{D}, \quad t \in [0,T), \tag{1}$$

where, for all $t \in [0,T)$, $p(z,t)$ is analytic in $\mathbb{D}$ with respect to z, $p(0,t) = 1$, $\operatorname{Re} p(z,t) > 0$ and $p(z,t)$ is measurable with respect to t for any $z \in \mathbb{D}$. A similar statement can be formulated for a decreasing chain of domains.

We consider a chain of bounded domains $\Omega(t)$, $0 \in \Omega(t)$, $\Omega(0) = \mathbb{D}$ with a boundary $\Gamma(t)$ and a chain $\Omega^*(t)$, $\infty \in \Omega^*(t)$, of unbounded domains with the same boundary $\Gamma(t)$. The method of Löwner-Kufarev evolution can be used to establish a connection between conformal radii of these domains. In [6], it is shown that if $\Omega(t)$ is decreasing, $p(\cdot,t) \in C^2(\overline{\mathbb{D}})$ for $t \in [0,T)$, $p(z,\cdot)$ is continuous in $[0,T)$ for $z \in \overline{\mathbb{D}}$ and $p(z,t)$, $p'(z,t)$ and $p''(z,t)$ are bounded in $\overline{\mathbb{D}} \times [0,T)$, then $\ln(r(\Omega^*(0),\infty)) = t + o(t)$, $t \to +0$.

We suppose now, that $\Gamma(t)$ is given by the polar equation $r = \gamma(\psi,t)$. Let $G(t)$, $0 \in G(t)$ be a chain of domains bounded by a curve with the polar equation $r = \gamma^{-1}(\psi,t)$. Let $f(z,t) = a(t)z + \ldots$ and $g(z,t) = b(t)z + \ldots$ conformally map $\mathbb{D}$ onto $\Omega(t)$ and $G(t)$, respectively, where $a(t) = r(\Omega(t),0)$, $b(t) = r(G(t),0)$ are positive, strictly monotone and continuous functions, $a(0) = b(0) = 1$. We can always choose the parameter t so that $a(t) = e^t$ ($a(t) = e^{-t}$ in the case of a decreasing chain of domains). The following theorem gives the asymptotic expansion for $b(t)$ in a neighbourhood of $t = 0$.

Theorem 1. *Let $\Omega(t)$, $t \in [0,T)$, be a chain of domains (increasing or decreasing), $0 \in \Omega(t)$, $r(\Omega(t),0) = e^{\pm t}$ for each $t \in [0,T)$, $\Omega(0) = \mathbb{D}$, and the boundary $\Gamma(t)$ for each $t \in [0,T)$ given by the polar equation $r = \gamma(\psi,t)$, $\psi \in [0,2\pi]$, where $\gamma \in C^{3+\alpha}$, $\alpha \in (0,1)$. Let $G(t)$ be a chain of domains bounded by the family of curves with the polar equation $r = \gamma_1(\psi,t) = (\gamma(\psi,t))^{-1}$. Then*

$$\log r(G(t),0) = \mp t + \left(\frac{1}{2\pi} \int_0^{2\pi} (\dot\gamma(\varphi,0))^2 - \ddot\gamma(\varphi,0)\, d\varphi \right) t^2 + o(t^2), \quad t \to +0. \tag{2}$$

By $\dot\gamma$, $\ddot\gamma$ we denote the first and second derivatives with respect to the parameter t, respectively. In general, we use the following convention. If f is a function of a real or complex variable and t is a parameter, then $\dot f$ denotes the derivative with respect to t, while f' denotes the derivative with respect to another variable.

It is not difficult to see that $r(\Omega^*(t),\infty) = r(G(t),0)$ for $t \in [0,T)$. So, we have the following corollary of Theorem 1, which is the main result of the article.

Corollary 1. *Let a chain of domains $\Omega(t)$ and their boundaries $\Gamma(t)$ be the same as in Theorem 1, and $\Omega^*(t)$ be a chain of the unbounded components of the complement* *of* $\Gamma(t)$. *Then*

$$\log r(\Omega^*(t),\infty) = \mp t + \left(\frac{1}{2\pi}\int\limits_0^{2\pi} (\dot\gamma(\varphi,0))^2 - \ddot\gamma(\varphi,0)\,d\varphi\right) t^2 + o(t^2), \quad t\to +0. \tag{3}$$

In Sect. 1 we deduce the integral representation for a driving function in the Löwner – Kufarev equation. We use it in Sect. 2, where Theorem 1 is proved.

1. LÖWNER – KUFAREV EQUATION

The following theorem gives the integral representation for the driving function $p(z,t)$ in the Löwner – Kufarev equation (1). Note that we do not suppose that $f'(0,t) = e^{\pm t}$, as it is usually done.

Theorem 2. *Let* $\Omega(t)$, $t\in[0,T)$ *be a chain of domains (increasing or decreasing),* $\Omega(0)=\mathbb{D}$, $0\in\Omega(t)$, *with a boundary* $\partial\Omega(t)$ *given by the polar equation* $r=\gamma(\psi,t)=1+\delta(\psi,t)$, $\psi\in[0,2\pi]$, *where* $\delta\in C^{3+\alpha}$, $\alpha\in(0,1)$. *Let* $f(z,t)=a(t)z+\ldots$ *conformally map* $\mathbb{D}$ *onto* $\Omega(t)$, $a(t)>0$. *Then* f *is differentiable with respect to* t *for* $t\in[0,T)$, $z\in\mathbb{D}$, *and satisfies the equation* (1) *where*

$$p(z,t) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1}{|f'(e^{i\varphi},t)|}\dot\delta(\psi(\varphi,t),t)\cos(\beta(\psi(\varphi,t),t))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \tag{4}$$

with $\psi(\varphi,t)=\arg f(e^{i\varphi},t)$ *and* $\beta(\psi,t) = -\arctan(\frac{\gamma'(\psi,t)}{\gamma(\psi,t)})$.

Remark 1. Note that $\beta(\psi,t)$ is an angle between a normal to the boundary $\partial\Omega(t)$ at the point $\gamma(\psi,t)e^{i\psi}$ and a radius vector of this point.

Remark 2. Here, the function $p(z,t)$ is analytic in $\mathbb{D}$ with respect to z, $\operatorname{Re} p(z,t)>0$ if $\Omega(t)$ is increasing and $\operatorname{Re} p(z,t)<0$ if $\Omega(t)$ is decreasing, $p(0,t)=\pm1$ if $a(t)=e^{\pm t}$.

Remark 3. Differentiating (1) with respect to z and putting $z=0$, we obtain an equation for the conformal radius $r(\Omega(t),0)=a(t)$

$$\frac{d}{dt}\log a(t) = p(0,t) = \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|f'(e^{i\varphi},t)|}\dot\delta(\psi(\varphi,t),t)\cos(\beta(\psi(\varphi,t),t))d\varphi.$$

First, we prove the following lemma.

Lemma 1. *Let* $\Omega_1\subset\Omega$ *be domains bounded by simple closed curves* Γ, Γ_1 *given by the polar equations* $r=\gamma(\psi)$, $r=\gamma_1(\psi)$, $\psi\in[0,2\pi]$, $\gamma,\gamma_1\in C^{3+\alpha}$, $\alpha\in(0,1)$. *Let* f *and* f_1 *conformally map* $\mathbb{D}$ *onto* Ω *and* Ω_1, *respectively,* $f(0)=f_1(0)=0$, $f'(0)>0$, $f_1'(0)>0$. *Let* $\delta(\psi)=\gamma(\psi)-\gamma_1(\psi)$ *satisfy* $|\delta(\psi)|<\varepsilon$, $|\delta'(\psi)|<\varepsilon$, $|\delta''(\psi)|<\varepsilon$. *Then*

$$f_1(z) = f\left(z\left(1-\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|f'(e^{i\varphi})|}\delta(\psi(\varphi))\cos\beta(\psi(\varphi))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi\right)\right)+ \\ +O(\varepsilon^2), \qquad z\in\mathbb{D}, \qquad \varepsilon\to+0, \tag{5}$$

with $\psi(\varphi)=\arg f(e^{i\varphi})$ *and* $\beta(\psi)=-\arctan(\frac{\gamma'(\psi)}{\gamma(\psi)})$.

We need the following theorem obtained by Siryk [12] (see also [13, p. 371]). It provides the asymptotic representation for functions conformally mapping $\mathbb{D}$ onto domains close to $\mathbb{D}$.

Theorem 3. *[12] Let Ω be a domain that contains 0 and is bounded by a curve given by the polar equation $r = 1 - \delta(\psi)$, $0 \leqslant \psi \leqslant 2\pi$, where ψ is twice differentiable and satisfies the conditions*

$$|\delta(\psi)| < \varepsilon, \quad |\delta'(\psi)| < \varepsilon, \quad |\delta''(\psi)| < \varepsilon.$$

Then a function $f : \mathbb{D} \to \Omega$, $f(0) = 0$, $f'(0) > 0$, mapping $\mathbb{D}$ conformally onto Ω has the asymptotic representation

$$f(z) = z\left(1 - \frac{1}{2\pi}\int_0^{2\pi} \delta(\psi)\frac{e^{i\psi}+z}{e^{i\psi}-z}\,d\psi\right) + O(\varepsilon^2), \quad \varepsilon \to +0. \tag{6}$$

Proof of Lemma 1. We denote the inverse function by $g = f^{-1}$. Since $\gamma \in C^{3+\alpha}$, f, f', f'', $f^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$ [14, p. 49] and f' does not vanish there [14, p. 48]. Hence g, g', g'', $g^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$.

The function g maps the curve Γ_1 onto a simple closed curve in $\mathbb{D}$, which has the following equation

$$\omega(\varphi) = g(\gamma_1(\psi(\varphi))e^{i\psi(\varphi)}) = g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)}), \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{7}$$

We have

$$\begin{gathered}\omega(\varphi) - e^{i\varphi} = g(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)}) - g(f(e^{i\varphi})) = \\ = -g'(f(e^{i\varphi}))\delta(\psi(\varphi))e^{i\psi(\varphi)} + O(\varepsilon^2) = -\frac{1}{f'(e^{i\varphi})}\delta(\psi(\varphi))e^{i\psi(\varphi)} + O(\varepsilon^2) = \\ = -\frac{1}{|f'(e^{i\varphi})|}\delta(\psi(\varphi))e^{i(\varphi-\beta(\psi(\varphi)))} + O(\varepsilon^2), \qquad \varepsilon \to +0,\end{gathered} \tag{8}$$

where $\beta(\psi(\varphi)) = \arg\frac{f'(e^{i\varphi})e^{i\varphi}}{f(e^{i\varphi})} = -\arctan\frac{\gamma'(\psi(\varphi))}{\gamma(\psi(\varphi))}$. Differentiating (7), we obtain

$$\begin{gathered}\omega'(\varphi) = g'(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[f'(e^{i\varphi})ie^{i\varphi} - \psi'(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi)))], \\ \omega''(\varphi) = g''(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[f'(e^{i\varphi})ie^{i\varphi} - \psi'(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi)))]^2 + \\ +g'(f(e^{i\varphi}) - \delta(\psi(\varphi))e^{i\psi(\varphi)})[-f''(e^{i\varphi})e^{2i\varphi} - f'(e^{i\varphi})e^{i\varphi} - \\ -\psi''(\varphi)e^{i\psi(\varphi)}(\delta'(\psi(\varphi)) + i\delta(\psi(\varphi))) - (\psi'(\varphi))^2e^{i\psi(\varphi)}(\delta''(\psi(\varphi)) + 2i\delta'(\psi(\varphi)) - \delta(\psi(\varphi)))].\end{gathered}$$

Since, g'', $g^{(3)}$ can be continuously extended to $\overline{\mathbb{D}}$ and $|\delta(\psi)| < \varepsilon$, $|\delta'(\psi)| < \varepsilon$, $|\delta''(\psi)| < \varepsilon$ we obtain the following estimates

$$\omega'(\varphi) = [g'(f(e^{i\varphi})) + O(\varepsilon)][f'(e^{i\varphi})ie^{i\varphi} + O(\varepsilon)] = ie^{i\varphi} + O(\varepsilon), \quad \varepsilon \to +0, \tag{9}$$

$$\omega''(\varphi) = [g''(f(e^{i\varphi})) + O(\varepsilon)][f'(e^{i\varphi})ie^{i\varphi} + O(\varepsilon)]^2 + [g'(f(e^{i\varphi})) + O(\varepsilon)][-f''(e^{i\varphi})e^{2i\varphi} -$$

$$-f'(e^{i\varphi})e^{i\varphi}+O(\varepsilon)]=\left[-\frac{f''(e^{i\varphi})}{(f'(e^{i\varphi}))^3}+O(\varepsilon)\right][-(f'(e^{i\varphi}))^2e^{2i\varphi}+O(\varepsilon)]+$$
$$+\left[\frac{1}{f'(e^{i\varphi})}+O(\varepsilon)\right][-f''(e^{i\varphi})e^{2i\varphi}-f'(e^{i\varphi})e^{i\varphi}+O(\varepsilon)]=$$
$$=\frac{f''(e^{i\varphi})}{f'(e^{i\varphi})}e^{2i\varphi}-\frac{f''(e^{i\varphi})}{f'(e^{i\varphi})}e^{2i\varphi}-e^{i\varphi}+O(\varepsilon)=-e^{i\varphi}+O(\varepsilon),\qquad \varepsilon\to+0. \tag{10}$$

Dividing (9) by $\omega(\varphi)$ gives

$$\frac{\partial}{\partial\varphi}\log\omega(\varphi)=\frac{ie^{i\varphi}+O(\varepsilon)}{e^{i\varphi}+O(\varepsilon)}=i+O(\varepsilon),\qquad \varepsilon\to+0, \tag{11}$$

therefore

$$\frac{\partial}{\partial\varphi}\arg\omega(\varphi)=1+O(\varepsilon),\qquad \varepsilon\to+0, \tag{12}$$

Hence, the curve $g(\Gamma_1)$ can be given by a polar equation for ε small enough. Denote $\mu(\varphi)=\arg\omega(\varphi)-\varphi$, $0\leqslant\varphi\leqslant 2\pi$. Using (8) we obtain

$$\mu(\varphi)=\arg\frac{g(f(e^{i\varphi})-\delta(\psi(\varphi))e^{i\psi(\varphi)})}{e^{i\varphi}}=\arg\left(1+\frac{g(f(e^{i\varphi})-\delta(\psi(\varphi))e^{i\psi(\varphi)})-e^{i\varphi}}{e^{i\varphi}}\right)=$$
$$=\arg\left(1-\frac{1}{|f'(e^{i\varphi})|}\delta(\psi(\varphi))e^{-i\beta(\psi(\varphi))}+O(\varepsilon^2)\right),\qquad \varepsilon\to+0,$$

Therefore, it is not difficult to see that $\mu(\varphi)=O(\varepsilon^2)$. From (12) we conclude that $\mu'(\varphi)=O(\varepsilon)$. Let $r=1-\Delta(\varphi)$ be the polar equation of $g(\Gamma_1)$. Then $\Delta(\varphi)=1-|\omega(\varphi_1)|$, where φ_1 is a unique solution of $\arg\omega(\varphi_1)=\varphi$. Hence $\mu(\varphi_1)+\varphi_1=\varphi$ and $\varphi_1=\varphi+O(\varepsilon^2)$. Applying (8) gives

$$\Delta(\varphi)=1-|\omega(\varphi_1)|=1-|\omega(\varphi+O(\varepsilon^2))|=1-|\omega(\varphi)+O(\varepsilon^2)|=$$
$$=1-\left|e^{i\varphi}+(\omega(\varphi)-e^{i\varphi})+O(\varepsilon^2)\right|=1-|e^{i\varphi}-\frac{1}{|f'(e^{i\varphi})|}\delta(\psi(\varphi))e^{i(\varphi-\beta(\psi(\varphi)))}+O(\varepsilon^2)|=$$
$$=1-|1-\frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|}e^{-i\beta(\psi(\varphi)))}+O(\varepsilon^2)|=$$
$$=\frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|}\cos\beta(\psi(\varphi))+O(\varepsilon^2),\quad 0\leqslant\varphi\leqslant 2\pi,\quad \varepsilon\to+0. \tag{13}$$

From (13) it easily follows that $|\Delta(\varphi)|=O(\varepsilon)$. Now we want to deduce similar estimates for $|\Delta'(\varphi)|$, $|\Delta''(\varphi)|$. It follows from (8)–(10) that $|\omega(\varphi)|'=O(\varepsilon)$, $|\omega(\varphi)|''=O(\varepsilon)$. Differentiating the equation $\Delta(\varphi+\mu(\varphi))=1-|\omega(\varphi)|$ we obtain

$$\Delta'(\varphi+\mu(\varphi))(1+\mu'(\varphi))=-|\omega(\varphi)|',$$

$$\Delta''(\varphi+\mu(\varphi))(1+\mu'(\varphi))^2+\Delta'(\varphi+\mu(\varphi))\mu''(\varphi)=-|\omega(\varphi)|''$$

Hence $|\Delta'(\varphi)|=O(\varepsilon)$, $|\Delta''(\varphi)|=O(\varepsilon)$.

Thus, the curve $g(\Gamma_1)$ has the polar equation $r=1-\Delta(\varphi)$ where $|\Delta(\varphi)|=O(\varepsilon)$, $|\Delta'(\varphi)|=O(\varepsilon)$, $|\Delta''(\varphi)|=O(\varepsilon)$, $\varepsilon\to+0$. Therefore, we can apply Theorem 3 for

$h = g \circ f_1$, $h(0) = 0$, $h'(0) > 0$, conformally mapping $\mathbb{D}$ onto the domain bounded by $g(\Gamma_1)$. Hence, (6) gives

$$h(z) = z(1 - \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\delta(\psi(\varphi))}{|f'(e^{i\varphi})|}\cos\beta(\psi(\varphi))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\varphi) + O(\varepsilon^2), \qquad z \in \mathbb{D}, \qquad \varepsilon \to +0.$$

Since $f_1 = f \circ h$, we obtain (5). $\square$

Proof of Theorem 2. Let $\Omega(t)$ be an increasing chain. Fix $t \in [0, T)$, h such that $t + h \in [0, T)$. First, let h be negative, so $\Omega(t + h) \in \Omega(t)$. Denote $\lambda(\psi, h) = \gamma(\psi, t) - \gamma(\psi, t + h)$. Since $\gamma \in C^{3+\alpha}$, we conclude that $|\lambda(\psi, h)| = O(h)$, $|\lambda'(\psi, h)| = O(h)$, $|\lambda''(\psi, h)| = O(h)$. So we can apply Lemma 1. By (5), we obtain

$$f(z, t + h) = f\left(z\left(1 - \int\limits_0^{2\pi} s(\varphi, z, t)\lambda(\psi(\varphi, t), h)d\varphi\right), t\right) + o(h) =$$

$$= f\left(z\left(1 + h\int\limits_0^{2\pi} s(\varphi, z, t)\dot{\delta}(\psi(\varphi, t), t)d\varphi\right), t\right) + o(h),$$

where

$$s(\varphi, z, t) = \frac{1}{2\pi}\frac{1}{|f'(e^{i\varphi}, t)|}\cos(\beta(\psi(\varphi, t), t))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}.$$

Therefore,

$$f(z, t + h) - f(z, t) = f(z + zh\int\limits_0^{2\pi} s(\varphi, z, t)\dot{\delta}(\psi(\varphi, t), t)d\varphi, t) - f(z, t) + o(h) =$$

$$= f'(z, t)zh\int\limits_0^{2\pi} s(\varphi, z, t)\dot{\delta}(\psi(\varphi, t), t)d\varphi + o(h), \qquad h \to 0.$$

Let now h be positive. Then $\Omega(t) \in \Omega(t + h)$, and $\lambda(\psi, h) = \gamma(\psi, t + h) - \gamma(\psi, t)$ satisfies all the conditions of Lemma 1. So we have

$$f(z, t) = f\left(z\left(1 - \int\limits_0^{2\pi} s(\varphi, z, t + h)\lambda(\psi(\varphi, t + h), h)d\varphi\right), t + h\right) + o(h) =$$

$$= f\left(z\left(1 - h\int\limits_0^{2\pi} s(\varphi, z, t + h)\dot{\delta}(\psi(\varphi, t + h), t)d\varphi\right), t + h\right) + o(h),$$

So we obtain

$$f(z, t + h) - f(z, t) = f(z, t + h) -$$

$$-f(z - zh\int\limits_0^{2\pi} s(\varphi, z, t + h)\dot{\delta}(\psi(\varphi, t + h), t)d\varphi, t + h) + o(h) =$$

$$= f'(z,t+h)zh\int\limits_0^{2\pi} s(\varphi,z,t+h)\dot\delta(\psi(\varphi,t+h),t)d\varphi + o(h), \qquad h\to 0,$$

Thus, we have shown that f is differentiable with respect to t and satisfies (1). One can similarly repeat the proof for a decreasing chain of domains. □

2. PROOF OF THEOREM 1

Let, for each $t\in[0,T)$, $f(\cdot,t)$ and $g(\cdot,t)$ conformally map $\mathbb{D}$ onto $\Omega(t)$ and $G(t)$, respectively, $f(0,t)=g(0,t)=0$, $f'(0,t)>0$, $g'(0,t)>0$. Denote $\delta(\psi,t)=\gamma(\psi,t)-1$, $\delta_1(\psi,t)=\gamma_1(\psi,t)-1$. By Theorem 2 and Remark 3, conformal radii satisfy the equations

$$\frac{d}{dt}\log r(\Omega(t),0)=p(0,t), \qquad \frac{d}{dt}\log r(G(t),0)=q(0,t), \tag{14}$$

where $p(z,t)$, $q(z,t)$ are given by

$$p(z,t)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|f'(e^{i\varphi},t)|}\dot\delta(\psi(\varphi,t),t)\cos(\beta(\psi(\varphi,t),t))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \tag{15}$$

$$q(z,t)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|g'(e^{i\varphi},t)|}\dot\delta_1(\psi_1(\varphi,t),t)\cos(\beta_1(\psi_1(\varphi,t),t))\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \tag{16}$$

with $\psi(\varphi,t)=\arg f(e^{i\varphi},t)$, $\beta(\psi,t)=-\arctan(\frac{\gamma'(\psi,t)}{\gamma(\psi,t)})$, $\psi_1(\varphi,t)=\arg g(e^{i\varphi},t)$, $\beta_1(\psi,t)=-\arctan(\frac{\gamma_1'(\psi,t)}{\gamma_1(\psi,t)})$. First, we want to prove the following equalities

$$p(z,0)=-q(z,0)=\frac{1}{2\pi}\int\limits_0^{2\pi}\dot\delta(\varphi,0)\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \qquad z\in\mathbb{D}, \tag{17}$$

$$\frac{\partial q(z,0)}{\partial t}-\frac{\partial p(z,0)}{\partial t}=\frac{1}{\pi}\int\limits_0^{2\pi}(\dot\delta(\varphi,0))^2-\ddot\delta(\varphi,0)\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \qquad z\in\mathbb{D}. \tag{18}$$

Elementary calculations lead us to the formulas

$$\dot\delta_1(\psi,0)=-\dot\delta(\psi,0), \tag{19}$$

$$\dot\delta_1'(\psi,0)=-\dot\delta'(\psi,0), \tag{20}$$

$$\ddot\delta_1(\psi,0)=-\ddot\delta(\psi,0)+2(\dot\delta(\psi,0))^2. \tag{21}$$

Since $f'(e^{i\varphi},0)=1$, $\psi(\varphi,0)=\varphi$, $\beta(\psi,0)=0$, representation (15) gives

$$p(z,0)=\frac{1}{2\pi}\int\limits_0^{2\pi}\dot\delta(\varphi,0)\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \qquad z\in\mathbb{D}.$$

Similarly we obtain

$$q(z,0)=\frac{1}{2\pi}\int\limits_0^{2\pi}\dot\delta_1(\varphi,0)\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi, \qquad z\in\mathbb{D}.$$

Thus, (19) gives (17).

Denote by $P(t)$ the expression under the integral in (15). Elementary calculations yield the following result

$$\dot{P}(0) = -\frac{\partial}{\partial t}(|f'(e^{i\varphi},t)|)\Big|_{t=0} \dot{\delta}(\varphi,0) + \dot{\delta}'(\varphi,0)\dot{\psi}(\varphi,0) + \ddot{\delta}(\varphi,0). \tag{22}$$

Similarly, we denote by $Q(t)$ the expression under the integral in (16) and obtain

$$\dot{Q}(0) = -\frac{\partial}{\partial t}(|g'(e^{i\varphi},t)|)\Big|_{t=0} \dot{\delta}_1(\varphi,0) + \dot{\delta}_1'(\varphi,0)\dot{\psi}_1(\varphi,0) + \ddot{\delta}_1(\varphi,0). \tag{23}$$

It easily follows from (1) that $\dot{\psi}(\varphi,0) = \operatorname{Im} p(e^{i\varphi},0)$, $\dot{\psi}_1(\varphi,0) = \operatorname{Im} q(e^{i\varphi},0)$. Therefore, (17) gives

$$\dot{\psi}_1(\varphi,0) = -\dot{\psi}(\varphi,0), \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{24}$$

Differentiating (1) with respect to z and putting $t=0$ we conclude that $\frac{\partial^2 f(z,t)}{\partial z \partial t}\Big|_{t=0} = p(z,0) + zp'(z,0)$. Hence

$$\dot{f}'(e^{i\varphi},0) = \frac{\partial}{\partial t} f'(e^{i\varphi},t)\Big|_{t=0} = p(e^{i\varphi},0) + e^{i\varphi}p'(e^{i\varphi},0).$$

Since

$$f'(e^{i\varphi},t) = f'(e^{i\varphi},0) + \dot{f}'(e^{i\varphi},0)t + o(t), \qquad t \to +0,$$

we see that

$$|f'(e^{i\varphi},t)| = |e^{i\varphi} + \dot{f}'(e^{i\varphi},0)t| + o(t) = \\ = 1 + |\dot{f}'(e^{i\varphi},0)| \cos(\arg(\dot{f}'(e^{i\varphi},0)) - \varphi)t + o(t), \qquad t \to +0.$$

Thus,

$$\frac{\partial}{\partial t}|f'(e^{i\varphi},t)|\Big|_{t=0} = |\dot{f}'(e^{i\varphi},0)| \cos(\arg(\dot{f}'(e^{i\varphi},0)) - \varphi), \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{25}$$

From (25) and (17) we deduce

$$\frac{\partial}{\partial t}|g'(e^{i\varphi},t)|\Big|_{t=0} = -\frac{\partial}{\partial t}|f'(e^{i\varphi},t)|\Big|_{t=0}, \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{26}$$

Formulas (19)–(24) and (26) show that $\dot{Q}(0) - \dot{P}(0) = -2\ddot{\delta}(\psi,0) + 2(\dot{\delta}(\psi,0))^2$, which leads to (18). One can deduce (2) from (14) and (17), (18). Indeed, let, for example, $\Omega(t)$ be an increasing chain of domains. Using (14) and (17) we obtain

$$\frac{d}{dt}\log r(G(t),0)\Big|_{t=0} = q(0,0) = -p(0,0) = -1,$$

Similarly, using (18) we obtain

$$\frac{d^2}{dt^2}\log r(G(t),0)\Big|_{t=0} = \frac{d}{dt}q(0,t)\Big|_{t=0} = \frac{1}{\pi}\int_0^{2\pi} (\dot{\delta}(\varphi,0))^2 - \ddot{\delta}(\varphi,0)\, d\varphi.$$

Acknowledgements: This work was supported by the Russian Science Foundation (project no. 17-11-01229).

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Cite this article as:
Zherdev A. V. An Asymptotic Relation for Conformal Radii of Two Nonoverlapping Domains. *Izv. Saratov Univ. (N. S.), Ser. Math. Mech. Inform.*, 2018, vol. 18, iss. 3, pp. 274–283. DOI: https://doi.org/10.18500/1816-9791-2018-18-3-274-283

УДК 517.54

АСИМПТОТИЧЕСКОЕ СООТНОШЕНИЕ ДЛЯ КОНФОРМНЫХ РАДИУСОВ ДВУХ НЕНАЛЕГАЮЩИХ ОБЛАСТЕЙ

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Жердев Андрей Владимирович, аспирант кафедры математического анализа, Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, Саратов, Астраханская, 83; Петрозаводский государственный университет, Россия, 185910, Республика Карелия, Петрозаводск, просп. Ленина, 33, jerdevandrey@gmail.com

В статье рассматривается семейство замкнутых жордановых кривых, заданных в полярной системе координат и непрерывно зависящих от параметра, и такое, что области, ограниченные этими кривыми, образуют возрастающее или убывающее семейство. Такое семейство областей описывается дифференциальным уравнением Левнера – Куфарева. Для рассмотренного случая получено интегральное представление для управляющей функции в этом уравнении. Используя это представление, получено асимптотическое соотношение, связывающее конформные радиусы ограниченной и неограниченной компоненты дополнения к жордановой кривой, когда ограниченная компонента близка к единичному кругу.

Ключевые слова: уравнение Левнера – Куфарева, конформный радиус, асимптотическое разложение, неналегающие области.

Благодарности. Работа выполнена при финансовой поддержке Российского научного фонда (проект № 17-11-01229).

Образец для цитирования:
Zherdev A. V. An Asymptotic Relation for Conformal Radii of Two Nonoverlapping Domains [Жердев А. В. Асимптотическое соотношение для конформных радиусов двух неналегающих областей] // Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика. 2018. Т. 18, вып. 3. С. 274–283. DOI: https://doi.org/10.18500/1816-9791-2018-18-3-274-283