



## On Inverse Problem for Differential Operators with Deviating Argument

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Second-order functional differential operators with a constant delay are considered. Properties of their spectral characteristics are obtained, and a nonlinear inverse spectral problem is studied, which consists in constructing operators from their spectra. We establish the uniqueness and develop a constructive procedure for solution of the inverse problem.

*Key words:* differential operators, deviating argument, inverse spectral problem.

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### INTRODUCTION

We study the inverse spectral problem for Sturm – Liouville differential operators with a constant delay. Such problems often appear in natural sciences and engineering (see, for example, monograph [1] and the references therein). Inverse spectral problems consist in constructing operators with given spectral characteristics. For the classical Sturm – Liouville operators the inverse problems have been studied fairly completely; the main results can be found in [2, 3]. However, differential operators with delay are essentially more difficult for investigating, since the main methods in the inverse problem theory (the transformation operator method and the method of spectral mappings [2, 3]) do not work for operators with delay. Note that some particular results on the inverse problems for operators with delay were obtained in [4–7].

Consider the boundary value problems  $L_j(q)$ ,  $j = 1, 2$ :

$$-y''(x) + q(x)y(x - a) = \lambda y(x), \quad x \in (0, \pi), \quad (1)$$

$$y(0) = y'(\pi) + H_j y(\pi) = 0, \quad (2)$$

where  $\lambda$  is the spectral parameter,  $a \in (0, \pi)$ ,  $q(x) \in L(a, \pi)$  is a complex-valued function, and  $q(x) \equiv 0$  for  $x \in [0, a]$ . We study the inverse problem of constructing the potential  $q(x)$  and the coefficients  $H_j$  from the given two spectra of the boundary value problems  $L_j(q)$ . More precisely, let  $\{\mu_{nj}\}_{n \geq 0}$ ,  $j = 1, 2$  be the eigenvalues of the problems  $L_j(q)$ .

**Inverse problem 1.** Given  $\{\mu_{nj}\}_{n \geq 0}$ ,  $j = 1, 2$ , construct  $q(x)$  and  $H_j$ .

We note that in the case of *large delay* when  $a \geq \pi/2$ , the characteristic functions of the problems  $L_j(q)$  depend on the potential  $q(x)$  linearly, i.e. the inverse problem becomes linear. This linear case was studied in [5, 7]. For  $a < \pi/2$  the characteristic functions depend on the potential nonlinearly, i.e. the inverse problem becomes nonlinear. This nonlinear case is seriously more difficult for investigating and for constructing the global solution of the inverse problem. In this paper we study namely nonlinear case. For definiteness, let  $a \in [2\pi/5, \pi/2)$ . The case  $a < 2\pi/5$  requires separate consideration. The main results of the paper are Theorem 1 and Algorithm 1, where a global constructive procedure for solving the inverse problem is provided, and the uniqueness of the solution is proved.



### 1. PROPERTIES OF SPECTRAL CHARACTERISTICS

Let  $S(x, \lambda)$  be the solution of Eq. (1) under the initial conditions  $S(0, \lambda) = 0$ ,  $S'(0, \lambda) = 1$ . Eigenvalues of the boundary value problem (1)–(2) coincide with the zeros of its characteristic function

$$p_j(\lambda) := S'(\pi, \lambda) + H_j S(\pi, \lambda), \quad j = 1, 2. \tag{3}$$

**Lemma 1.** *Boundary value problem  $L_j$  has a countable set of eigenvalues  $\{\mu_{nj}\}_{n \geq 0}$ , and for  $n \rightarrow \infty$ :*

$$\sqrt{\mu_{nj}} = \left(n + \frac{1}{2}\right) + \frac{A_0 \cos(n + 1/2)a}{2\pi n} + \frac{H_j}{\pi n} + o\left(\frac{1}{n}\right), \tag{4}$$

where  $A_0 = \int_a^\pi q(t) dt$ .

**Lemma 2.** *The specification of the spectrum  $\{\mu_{nj}\}_{n \geq 0}$  uniquely determines  $p_j(\lambda)$  via*

$$p_j(\lambda) = \prod_{n=0}^{\infty} \frac{\mu_{nj} - \lambda}{(n + 1/2)^2}. \tag{5}$$

Let us study the connections between the characteristic functions  $p_j(\lambda)$  and the potential  $q(x)$ . Let  $\lambda = \rho^2$ . The function  $S(x, \lambda)$  satisfies the integral equation

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_a^x \frac{\sin \rho(x-t)}{\rho} q(t) S(t-a, \lambda) dt. \tag{6}$$

Solving (6) we get for  $x \geq 2a$ :

$$S(x, \lambda) = S_0(x, \lambda) + S_1(x, \lambda) + S_2(x, \lambda),$$

where

$$S_0(x, \lambda) = \frac{\sin \rho x}{\rho}, \tag{7}$$

$$S_1(x, \lambda) = \int_a^x \frac{\sin \rho(x-t)}{\rho} q(t) S_0(t-a, \lambda) dt, \tag{8}$$

$$S_2(x, \lambda) = \int_{2a}^x \frac{\sin \rho(x-t)}{\rho} q(t) S_1(t-a, \lambda) dt, \tag{9}$$

Using (7) and (8), we calculate

$$S_1(x, \lambda) = -\frac{\cos \rho(x-a)}{2\rho^2} \int_a^x q(t) dt + \frac{1}{2\rho^2} \int_a^x q(t) \cos \rho(x-2t+a) dt. \tag{10}$$

Denote  $\Delta_k(\lambda) := S^{(k)}(\pi, \lambda)$ ,  $k = 0, 1$ . In view of (3), one has

$$p_j(\lambda) := \Delta_1(\lambda) + H_j \Delta_0(\lambda), \quad j = 1, 2, \tag{11}$$



and consequently,

$$\Delta_0(\lambda) = \frac{1}{H_1 - H_2} (p_1(\lambda) - p_2(\lambda)). \tag{12}$$

Taking (7), (9) and (10) into account, we obtain

$$\Delta_0(\lambda) = \frac{\sin \rho \pi}{\rho} - A_0 \frac{\cos \rho(\pi - a)}{2\rho^2} + \frac{1}{2\rho^2} \int_a^\pi q(t) \cos \rho(2t - \pi - a) dt + S_2(\pi, \lambda),$$

$$\Delta_1(\lambda) = \cos \rho \pi + A_0 \frac{\sin \rho(\pi - a)}{2\rho} + \frac{1}{2\rho} \int_a^\pi q(t) \sin \rho(2t - \pi - a) dt + S'_2(\pi, \lambda).$$

Denote

$$\Delta_0^*(\rho) := 2\rho^2 \left( \Delta_0(\lambda) - \frac{\sin \rho \pi}{\rho} + A_0 \frac{\cos \rho(\pi - a)}{2\rho^2} \right), \tag{13}$$

$$\Delta_1^*(\rho) := 2\rho \left( \Delta_1(\lambda) - \cos \rho \pi - A_0 \frac{\sin \rho(\pi - a)}{2\rho} \right). \tag{14}$$

Then

$$\Delta_0^*(\rho) = \int_a^\pi q(t) \cos \rho(2t - \pi - a) dt + \delta_0(\rho), \tag{15}$$

$$\Delta_1^*(\rho) = \int_a^\pi q(t) \sin \rho(2t - \pi - a) dt + \delta_1(\rho), \tag{16}$$

where  $\delta_0(\rho) = 2\rho^2 S_2(\pi, \lambda)$ ,  $\delta_1(\rho) = 2\rho S'_2(\pi, \lambda)$ . Using (9) and (10), we infer

$$2\rho \delta_0(\rho) = -A \sin \rho(\pi - 2a) + \frac{1}{2} \int_{-(\pi-2a)}^{(\pi-2a)} Q(\xi) \sin \rho \xi d\xi, \tag{17}$$

$$2\rho \delta_1(\rho) = -A \cos \rho(\pi - 2a) - \frac{1}{2} \int_{-(\pi-2a)}^{(\pi-2a)} Q(\xi) \cos \rho \xi d\xi, \tag{18}$$

where

$$A = \int_{2a}^\pi q(t) dt \int_a^{t-a} q(s) ds,$$

$$Q(\xi) = Q_1(\xi/2 + \pi/2 + a) - Q_2(\xi/2 + \pi/2) - Q_3(\xi/2 + \pi/2),$$

$$Q_1(x) = q(x) \int_a^{x-a} q(s) ds, \quad Q_2(x) = q(x) \int_{x+a}^\pi q(s) ds, \quad Q_3(x) = \int_{x+a}^\pi q(s) q(s-x) ds.$$

For simplicity we assume that  $q(x) \in AC[a, \pi]$ . The general case requires small technical modifications. Denote  $q_1(x) := q'(x)$ . Taking (15)–(18) into account, we get

$$\begin{aligned} 4\rho \Delta_0^*(\rho) &= B_1 \sin \rho(\pi - a) - 2A \sin \rho(\pi - 2a) - \\ &- \int_{-(\pi-a)}^{(\pi-a)} q_0(\xi) \sin \rho \xi d\xi + \int_{-(\pi-2a)}^{(\pi-2a)} Q(\xi) \sin \rho \xi d\xi, \\ 4\rho \Delta_1^*(\rho) &= B_2 \cos \rho(\pi - a) - 2A \cos \rho(\pi - 2a) + \end{aligned} \tag{19}$$



$$+ \int_{-(\pi-a)}^{(\pi-a)} q_0(\xi) \cos \rho \xi \, d\xi - \int_{-(\pi-2a)}^{(\pi-2a)} Q(\xi) \cos \rho \xi \, d\xi, \tag{20}$$

where  $B_1 = 2(q(a) + q(\pi))$ ,  $B_2 = 2(q(a) - q(\pi))$ ,  $q_0(\xi) = q_1(\xi/2 + \pi/2 + a/2)$ . Denote

$$d_0(\rho) = 4\rho\Delta_0^*(\rho) - B_1 \sin \rho(\pi - a) + 2A \sin \rho(\pi - 2a), \tag{21}$$

$$d_1(\rho) = 4\rho\Delta_1^*(\rho) - B_2 \cos \rho(\pi - a) + 2A \cos \rho(\pi - 2a). \tag{22}$$

It follows from (19)–(20) and (21)–(22) that

$$d_0(\rho) = - \int_{-(\pi-a)}^{(\pi-a)} R(\xi) \sin \rho \xi \, d\xi, \quad d_1(\rho) = \int_{-(\pi-a)}^{(\pi-a)} R(\xi) \cos \rho \xi \, d\xi, \tag{23}$$

where  $R(\xi) = q_0(\xi) - Q(\xi)$ , and  $Q(\xi) \equiv 0$  outside the interval  $(-\pi - 2a, \pi - 2a)$ . In particular, this yields

$$q_1(x) = R(2x - \pi - a) + Q_1(x + a/2) - Q_2(x - a/2) - Q_3(x - a/2), \quad x \in (3a/2, \pi - a/2). \tag{24}$$

Denote by  $\{\lambda_n\}_{n \geq 1}$  the zeros of the entire function  $\Delta_0(\lambda)$ . Then

$$\sqrt{\lambda_n} = n + \frac{A_0 \cos na}{2\pi n} + o\left(\frac{1}{n}\right). \tag{25}$$

## 2. SOLUTION OF THE INVERSE PROBLEM

In this section we present our main results: a constructive procedure for solving the inverse problem and the corresponding uniqueness theorem. The solution of Inverse problem 1 can be found by the following algorithm.

**Algorithm 1.** Let the spectra  $\{\mu_{nj}\}_{n \geq 0}$ ,  $j = 1, 2$  be given.

1. Construct  $p_j(\lambda)$ ,  $j = 0, 1$ , via (5).
2. Find  $H_1 - H_2$ , using (4):

$$H_1 - H_2 = \pi \lim_{n \rightarrow \infty} (\sqrt{\mu_{n1}} - \sqrt{\mu_{n2}}).$$

3. Construct  $\Delta_0(\lambda)$ , using (12), and calculate  $\lambda_n$ .
4. Find  $A_0$  from (25).
5. Calculate  $H_1$  and  $H_2$ , using (4).
6. Find  $\Delta_1(\lambda)$  from (11).
7. Construct  $\Delta_j^*(\lambda)$ ,  $j = 0, 1$ , according to (13) and (14).
8. Calculate  $A, B_1$  and  $B_2$ , using (19) and (12), and find

$$q(a) = (B_1 + B_2)/4, \quad q(\pi) = (B_1 - B_2)/4.$$

9. Construct  $d_0(\rho)$  and  $d_1(\rho)$  by (21) and (22).
10. Calculate  $R(\xi)$ , using (23).
11. Find  $q_0(\xi)$  for  $\xi \in (-\pi - a, -(\pi - 2a)) \cup (\pi - 2a, \pi - a)$ :  $q_0(\xi) = R(\xi)$ .
12. Calculate  $q_1(x) = q_0(2x - \pi - a)$  for  $x \in (a, 3a/2) \cup (\pi - a/2, \pi)$ .



13. Find

$$q(x) = q(a) + \int_a^x q_1(t) dt, \quad x \in (a, 3a/2), \quad q(x) = q(\pi) - \int_x^\pi q_1(t) dt, \quad x \in (\pi - a/2, \pi).$$

14. Using (24) and knowledge of  $q(x)$  for  $x \in (a, 3a/2) \cup (\pi - a/2, \pi)$ , construct  $q_1(x)$  for  $x \in (3a/2, \pi - a/2)$ :

$$q_1(x) = R(2x - \pi - a) + q(x + a/2) \int_a^{x-a/2} q(s) ds - q(x - a/2) \int_{x+a/2}^\pi q(s) ds - \int_{x+a/2}^\pi q(s)q(s - x + a/2) ds.$$

15. Calculate  $q(x)$  for  $x \in (3a/2, \pi - a/2)$ .

Thus, the following theorem is proved.

**Theorem 1.** *The specification of two spectra  $\{\mu_{nj}\}_{n \geq 0}$ ,  $j = 1, 2$  uniquely determines the potential  $q(x)$  and the coefficients  $H_1, H_2$ . The solution of Inverse problem 1 can be found by Algorithm 1.*

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## ОБ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

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Рассматриваются функционально-дифференциальные операторы второго порядка с постоянным запаздыванием. Установлены свойства их спектральных характеристик и исследуется нелинейная обратная спектральная задача, которая состоит в построении операторов по их спектрам. Доказана единственность решения обратной задачи и указана конструктивная процедура ее решения.

*Ключевые слова:* дифференциальные операторы, отклоняющийся аргумент, обратная спектральная задача.

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