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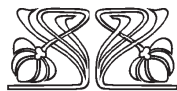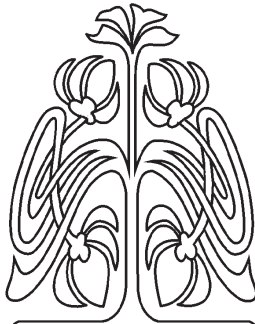
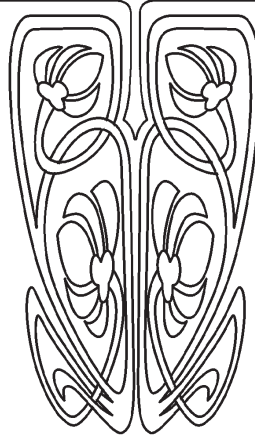## Approximation of Continuous 2π-Periodic Piecewise Smooth Functions by Discrete Fourier Sums

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Gasan G. Akniyev, https://orcid.org/0000-0001-8533-4277, Dagestan Scientific Center RAS, 45 M. Gadzhieva St., 367025 Makhachkala, Russia, hasan.akniyev@gmail.com

Let N be a natural number greater than 1. Select N uniformly distributed points $t_k = 2\pi k/N + u$ ($0 \leqslant k \leqslant N - 1$), and denote by $L_{n,N}(f) = L_{n,N}(f, x)$ ($1 \leqslant n \leqslant N/2$) the trigonometric polynomial of order n possessing the least quadratic deviation from f with respect to the system $\{t_k\}_{k=0}^{N-1}$. Select $m + 1$ points $-\pi = a_0 < a_1 < \ldots < a_{m-1} < a_m = \pi$, where $m \geqslant 2$, and denote $\Omega = \{a_i\}_{i=0}^m$. Denote by C_Ω^r a class of 2π-periodic continuous functions f, where f is r-times differentiable on each segment $\Delta_i = [a_i, a_{i+1}]$ and $f^{(r)}$ is absolutely continuous on Δ_i. In the present article we consider the problem of approximation of functions $f \in C_\Omega^2$ by the polynomials $L_{n,N}(f, x)$. We show that instead of the estimate $|f(x) - L_{n,N}(f, x)| \leqslant c \ln n/n$, which follows from the well-known Lebesgue inequality, we found an exact order estimate $|f(x) - L_{n,N}(f, x)| \leqslant c/n$ ($x \in \mathbb{R}$) which is uniform with respect to n ($1 \leqslant n \leqslant N/2$). Moreover, we found a local estimate $|f(x) - L_{n,N}(f, x)| \leqslant c(\varepsilon)/n^2$ ($|x - a_i| \geqslant \varepsilon$) which is also uniform with respect to n ($1 \leqslant n \leqslant N/2$). The proofs of these estimations are based on comparing of approximating properties of discrete and continuous finite Fourier series.

Keywords: function approximation, trigonometric polynomials, Fourier series.

INTRODUCTION

We begin by establishing some notations. Let Ω be a set of $m + 1$ points $\{a_i\}_{i=0}^m$ ($m > 2$) such that $-\pi = a_0 < < a_1 < \ldots < a_{m-1} < a_m = \pi$. We denote by $C_\Omega^{0,r}$ the class of 2π-periodic functions with r absolutely continuous derivatives on each interval (a_i, a_{i+1}) and by C_Ω^r the subclass of

all continuous functions in $C_\Omega^{0,r}$ (here we say that a function f is absolutely continuous on an interval (a, b) if the function $\overline{f}$ is absolutely continuous on the segment $[a, b]$, where $\overline{f}(x) = f(x)$ for $x \in (a, b)$, $\overline{f}(a) = f(a + 0)$, and $\overline{f}(b) = f(b - 0)$).

We denote by $L_{n,N}(f, x)$ a trigonometric polynomial of order n possessing the least quadratic deviation from the function f at the points $\{t_j\}_{i=0}^{N-1}$, where $t_j = u + 2\pi j/N$, $n \leqslant N/2$, $N \geqslant 2$, and $u \in \mathbb{R}$. In other words, $L_{n,N}(f, x)$ provides the minimum for the sum $\sum_{j=0}^{N-1} |f(t_j) - T_n(t_j)|^2$ on the set of all trigonometric polynomials of order at most n. To read more about function approximation by trigonometric polynomials see [1–10].

Also, in this paper we denote by c or $c(b_1, b_2, \ldots, b_k)$ some positive constants, which depend only on specified arguments (if any) and may vary from line to line, and by $S_n(f, x)$ the n-th partial sum of the Fourier series of the function f. We also note that it is easy to show that the Fourier series of any $f \in C_\Omega^2$ converges uniformly on $\mathbb{R}$ and the following representation is possible:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \tag{1}$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt. \tag{2}$$

The goal of this work is to estimate the value $|f(x) - L_{n,N}(f, x)|$ for $f \in C_\Omega^2$. Note that the special case of this problem is considered in [11], where the value $|f(x) - L_{n,N}(f, x)|$ is estimated for a 2π-periodic function $f(x) = |x|$ ($x \in [-\pi, \pi]$). In this work, we generalize the results from [11] for any function $f \in C_\Omega^2$, as stated in the following theorem:

Theorem 1. *For $f \in C_\Omega^2$ the following inequalities hold:*

$$|f(x) - L_{n,N}(f, x)| \leqslant \frac{c(f)}{n}, \quad x \in \mathbb{R}, \tag{3}$$

$$|f(x) - L_{n,N}(f, x)| \leqslant \frac{c(f, \varepsilon)}{n^2}, \quad x \in |x - a_i| \geqslant \varepsilon. \tag{4}$$

The order of these estimates cannot be improved.

To prove this theorem we use a lemma from [12]:

Lemma 1 (Sharapudinov, [12]). *If the Fourier series of f converges at the points $t_k = u + 2k\pi/N$, then the representation*

$$L_{n,N}(f, x) = S_n(f, x) + R_{n,N}(f, x), \tag{5}$$

where

$$R_{n,N}(f, x) = \frac{2}{\pi} \sum_{\mu=1}^{\infty} \int_{-\pi}^{\pi} D_n(x - t) \cos \mu N(u - t) f(t) dt, \tag{6}$$

holds true, where $2n < N$ and $D_n(x)$ is the Dirichlet kernel:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx. \tag{7}$$

This lemma considers only the case $2n < N$. If $2n = N$ (when N is even) we can write (see [12])

$$L_{n,2n}(f, x) = L_{n-1,2n}(f, x) + a_n^{(2n)}(f) \cos n(x - u), \qquad (8)$$

where

$$a_n^{(2n)}(f) = \frac{1}{2n} \sum_{k=0}^{2n-1} f(t_k) \cos n(t_k - u). \qquad (9)$$

To prove the inequalities (3) and (4) from Theorem 1 we use the formulas

$$|f(x) - L_{n,N}(f, x)| \leqslant |f(x) - S_n(f, x)| + |R_{n,N}(f, x)|, \quad n < N/2, \qquad (10)$$

$$|f(x) - L_{n,N}(f, x)| \leqslant |f(x) - S_{n-1}(f, x)| + |R_{n-1,N}(f, x)| + \left| a_n^{(2n)}(f) \right|, \quad n = N/2, \quad (11)$$

which immediately follow from (5) and (8).

The estimates for the values $|f(x) - S_n(f, x)|$, $|R_{n,N}(f, x)|$, and $|a_n^{(2n)}(f)|$ are found in the following sections.

1. THE ESTIMATE FOR $|f(x) - S_n(f, x)|$

To estimate the value $|f(x) - S_n(f, x)|$ we need the following lemma.

Lemma 2. *For $f \in C_\Omega^2$ the following inequality holds:*

$$\left| \int_{-\pi}^{\pi} f(t) h_p(k(t + \alpha)) dt \right| \leqslant \frac{c(f)}{k^2},$$

where $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and

$$h_p(x) = \begin{cases} \cos x, & p = 0, \\ \sin x, & p = 1. \end{cases} \qquad (12)$$

Proof. Performing integration by parts two times we have

$$\int_{-\pi}^{\pi} f(t) h_p(k(t + \alpha)) dt = \frac{(-1)^{p+1}}{k} \int_{-\pi}^{\pi} f'(t) h_{1-p}(k(t + \alpha)) dt =$$

$$= \frac{1}{k^2} \left[\sum_{i=0}^{m-1} \left(f'(a_i - 0) - f'(a_i + 0) \right) h_p(k(a_i + \alpha)) - \int_{-\pi}^{\pi} f''(t) h_p(k(t + \alpha)) dt \right].$$

From this we can get the estimate

$$\left| \int_{-\pi}^{\pi} f(t) h_p(k(t + \alpha)) dt \right| \leqslant \frac{1}{k^2} \left[\sum_{i=0}^{m-1} \left| f'(a_i - 0) - f'(a_i + 0) \right| + \int_{-\pi}^{\pi} \left| f''(t) \right| dt \right] \leqslant \frac{c(f)}{k^2}.$$

$\square$

Lemma 3. *For* $f \in C_\Omega^2$ *the following inequalities hold:*

$$|f(x) - S_n(f,x)| \leqslant \frac{c(f)}{n}, \quad x \in \mathbb{R}, \tag{13}$$

$$|f(x) - S_n(f,x)| \leqslant \frac{c(f,\varepsilon)}{n^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{14}$$

Proof. Here we prove only (13) because the proof for inequality (14) can be found in [13, Theorem 2.1]. Using (1) and (2) we can write

$$f(x) - S_n(f,x) = \sum_{k=n+1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right).$$

Applying Lemma 2 to (2) we get $|a_k| \leqslant c(f)/k^2$ and $|b_k| \leqslant c(f)/k^2$, which gives us

$$|f(x) - S_n(f,x)| \leqslant \sum_{k=n+1}^{\infty} \left(|a_k| + |b_k| \right) \leqslant c(f)/n.$$

$\square$

2. THE ESTIMATE FOR $|R_{n,N}(f,x)|$

From (6) and (7) follows $R_{n,N}(f,x) = R_{n,N}^1(f,x) + R_{n,N}^2(f,x)$, where

$$R_{n,N}^1(f,x) = \frac{1}{\pi} \sum_{\mu=1}^{\infty} \int_{-\pi}^{\pi} f(t) \cos \mu N(u - t) dt,$$

$$R_{n,N}^2(f,x) = \frac{2}{\pi} \sum_{\mu=1}^{\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \cos k(x - t) \cos \mu N(u - t) dt. \tag{15}$$

Obviously, $|R_{n,N}(f,x)| \leqslant \left| R_{n,N}^1(f,x) \right| + \left| R_{n,N}^2(f,x) \right|$. The values $\left| R_{n,N}^1(f,x) \right|$ and $\left| R_{n,N}^2(f,x) \right|$ are estimated later in this section, but first we prove three auxiliary lemmas.

Lemma 4. *For* $f \in C_\Omega^{0,1}$ *the following holds:*

$$\int_{-\pi}^{\pi} f(t) h_p(k(t-x)) h_q(\mu N(t-u)) dt =$$

$$= \frac{(-1)^q \mu N}{(\mu N)^2 - k^2} \sum_{i=0}^{m-1} \left(f(a_i - 0) - f(a_i + 0) \right) h_p(k(a_i - x)) h_{1-q}(\mu N(a_i - u)) -$$

$$- \frac{(-1)^q \mu N}{(\mu N)^2 - k^2} \int_{-\pi}^{\pi} f'(t) h_p(k(t-x)) h_{1-q}(\mu N(t-u)) dt +$$

$$+ \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2} \sum_{i=0}^{m-1} \left(f(a_i - 0) - f(a_i + 0) \right) h_{1-p}(k(a_i - x)) h_q(\mu N(a_i - u)) -$$

$$- \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2} \int_{-\pi}^{\pi} f'(t) h_{1-p}(k(t-x)) h_q(\mu N(t-u)) dt. \tag{16}$$

Proof. Perform integration by parts:

$$\int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N} \sum_{i=0}^{m-1} \left(f(a_i-0) - f(a_i+0)\right) h_p(k(a_i-x))h_{1-q}(\mu N(a_i-u))-$$

$$- \frac{(-1)^q}{\mu N} \int\limits_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt+$$

$$+ \frac{(-1)^{p+q}k}{\mu N} \int\limits_{-\pi}^{\pi} f(t)h_{1-p}(k(t-x))h_{1-q}(\mu N(t-u))dt. \tag{17}$$

Repeat integration by parts for the last integral in (17):

$$\int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N} \sum_{i=0}^{m-1} \left(f(a_i-0) - f(a_i+0)\right) h_p(k(a_i-x))h_{1-q}(\mu N(a_i-u))-$$

$$- \frac{(-1)^q}{\mu N} \int\limits_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt+$$

$$+ \frac{(-1)^{1+p}k}{(\mu N)^2} \sum_{i=0}^{m-1} \left(f(a_i-0) - f(a_i+0)\right) h_{1-p}(k(a_i-x))h_q(\mu N(a_i-u))-$$

$$- \frac{(-1)^{1+p}k}{(\mu N)^2} \int\limits_{-\pi}^{\pi} f'(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt+$$

$$+ \frac{k^2}{(\mu N)^2} \int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt.$$

By moving the last integral from the right side to the left and dividing both sides by $\frac{(\mu N)^2 - k^2}{(\mu N)^2}$ we get (16). $\qquad\square$

Corollary 1. *If $f \in C^2_\Omega$, then $f(a_i-0) - f(a_i+0) = 0$, so we can write (16) as*

$$\int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt = \frac{(-1)^q \mu N}{(\mu N)^2 - k^2} \int\limits_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt-$$

$$- \frac{(-1)^{1+p}k}{(\mu N)^2 - k^2} \int\limits_{-\pi}^{\pi} f'(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt.$$

Lemma 5. *The following estimate holds:*

$$\left| \sum_{k=1}^{n} h_p(kx) \right| \leqslant \frac{1}{\left| \sin \frac{x}{2} \right|}.$$

Proof. The proof is obvious and follows from well-known formulas

$$\sum_{k=1}^{n} \sin(kx) = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}, \quad \sum_{k=1}^{n} \cos(kx) = \frac{\cos\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

$\square$

Lemma 6. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a monotonous sequence (either increasing or decreasing) of n positive numbers. The following holds:*

$$\left| \sum_{k=1}^{n} \alpha_k h_p(kx) \right| \leqslant \frac{2\alpha_n + \alpha_1}{\left| \sin \frac{x}{2} \right|}.$$

Proof. After performing Abel transformation (summation by parts) we have:

$$\sum_{k=1}^{n} \alpha_k h_p(kx) = \alpha_n \sum_{j=1}^{n} h_p(jx) - \sum_{k=1}^{n-1} (\alpha_{k+1} - \alpha_k) \sum_{j=1}^{k} h_p(jx).$$

Using Lemma 5 and the fact that $\sum_{k=1}^{n} |\alpha_{k+1} - \alpha_k| = |\sum_{k=1}^{n} \alpha_{k+1} - \alpha_k|$ we can write

$$\left| \sum_{k=1}^{n} \alpha_k h_p(kx) \right| \leqslant \alpha_n \left| \sum_{j=1}^{n} h_p(jx) \right| + \sum_{k=1}^{n-1} |\alpha_{k+1} - \alpha_k| \left| \sum_{j=1}^{k} h_p(jx) \right| \leqslant$$

$$\leqslant \frac{1}{\left| \sin \frac{x}{2} \right|} \left(\alpha_n + \left| \sum_{k=1}^{n-1} \alpha_{k+1} - \alpha_k \right| \right) \leqslant \frac{2\alpha_n + \alpha_1}{\left| \sin \frac{x}{2} \right|}.$$

$\square$

Lemma 7. *The following inequality holds:* $\left| R_{n,N}^1(f,x) \right| \leqslant c(f)/N^2.$

Proof. Using Lemma 2 we have

$$\left| R_{n,N}^1(f,x) \right| \leqslant \frac{1}{\pi} \sum_{\mu=1}^{\infty} \left| \int_{-\pi}^{\pi} f(t) \cos \mu N(u-t) dt \right| \leqslant \frac{c(f)}{N^2} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \leqslant \frac{c(f)}{N^2}.$$

$\square$

Lemma 8. *The following estimates hold:*

$$\left| R_{n,N}^2(f,x) \right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \tag{18}$$

$$\left| R_{n,N}^2(f,x) \right| \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{19}$$

Proof. Rewrite (15) using (12):

$$R_{n,N}^2(f,x) = \frac{2}{\pi} \sum_{\mu=1}^{\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) h_0(k(t-x)) h_0(\mu N(t-u)) dt.$$

Using Corollary 1 we rewrite the above formula as follows:

$$R_{n,N}^2(f,x) = \frac{2}{\pi N} \sum_{\mu=1}^{\infty} \frac{1}{\mu} \sum_{k=1}^{n} \frac{1}{1-\left(\frac{k}{\mu N}\right)^2} \int_{-\pi}^{\pi} f'(t) h_0(k(t-x)) h_1(\mu N(t-u)) dt +$$

$$+\frac{2}{\pi N} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \sum_{k=1}^{n} \frac{k}{N} \frac{1}{1-\left(\frac{k}{\mu N}\right)^2} \int_{-\pi}^{\pi} f'(t) h_1(k(t-x)) h_0(\mu N(t-u)) dt =$$

$$= R_{n,N}^{2.1}(f,x) + R_{n,N}^{2.2}(f,x).$$

For brevity we only consider here estimation of $\left|R_{n,N}^{2.1}(f,x)\right|$ because $\left|R_{n,N}^{2.2}(f,x)\right|$ can be estimated in almost the same way. Obviously, $f' \in C_\Omega^{0,1}$, so we can apply Lemma 4 to $R_{n,N}^{2.1}(f,x)$:

$$R_{n,N}^{2.1}(f,x) = \frac{-2}{\pi N^2} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \sum_{k=1}^{n} \frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2} \sum_{i=0}^{m-1} \left(f'(a_i-0) - f'(a_i+0)\right) \times$$

$$\times h_0(k(a_i-x)) h_0(\mu N(a_i-u)) +$$

$$+\frac{2}{\pi N^2} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \sum_{k=1}^{n} \frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2} \int_{-\pi}^{\pi} f''(t) h_0(k(t-x)) h_0(\mu N(t-u)) dt +$$

$$+\frac{-2}{\pi N^3} \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} \sum_{k=1}^{n} \frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2} \sum_{i=0}^{m-1} \left(f'(a_i-0) - f'(a_i+0)\right) \times$$

$$\times h_1(k(a_i-x)) h_1(\mu N(a_i-u)) +$$

$$+\frac{2}{\pi N^3} \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} \sum_{k=1}^{n} \frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2} \int_{-\pi}^{\pi} f''(t) h_1(k(t-x)) h_1(\mu N(t-u)) dt =$$

$$= R_{n,N}^{2.1.1}(f,x) + R_{n,N}^{2.1.2}(f,x) + R_{n,N}^{2.1.3}(f,x) + R_{n,N}^{2.1.4}(f,x).$$

Begin with $R_{n,N}^{2.1.2}(f,x)$. Applying Lemma 4 we get

$$R_{n,N}^{2.1.2}(f,x) = \frac{2}{\pi N^3} \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} \sum_{k=1}^{n} \frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3} \sum_{i=0}^{m-1} \left(f''(a_i-0) - f''(a_i+0)\right) \times$$

$$\times h_0(k(a_i-x)) h_1(\mu N(a_i-u)) +$$

$$+\frac{-2}{\pi N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int_{-\pi}^{\pi}f'''(t)h_0(k(t-x))h_1(\mu N(t-u))dt+$$

$$+\frac{-2}{\pi N^4}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left(f''(a_i-0)-f''(a_i+0)\right)\times$$

$$\times h_1(k(a_i-x))h_0(\mu N(a_i-u))+$$

$$+\frac{2}{\pi N^4}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int_{-\pi}^{\pi}f'''(t)h_1(k(t-x))h_0(\mu N(t-u))dt.$$

From this we can get the estimate

$$\left|R_{n,N}^{2.1.2}(f,x)\right|\leqslant\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left|f''(a_i-0)-f''(a_i+0)\right|+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int_{-\pi}^{\pi}\left|f'''(t)\right|dt+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k/N}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left|f''(a_i-0)-f''(a_i+0)\right|+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k/N}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int_{-\pi}^{\pi}\left|f'''(t)\right|dt\leqslant\frac{c(f)}{N^2}.$$

In the same way we can get $\left|R_{n,N}^{2.1.4}(f,x)\right|\leqslant c(f)/N^2$. Now we consider $\left|R_{n,N}^{2.1.1}(f,x)\right|$ and $\left|R_{n,N}^{2.1.3}(f,x)\right|$. We will estimate here only $\left|R_{n,N}^{2.1.1}(f,x)\right|$ because the other one can be estimated in the similar way. After a simple transformation we have

$$R_{n,N}^{2.1.1}(f,x)=\frac{-2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{i=0}^{m-1}\left(f'(a_i-0)-f'(a_i+0)\right)h_0(\mu N(a_i-u))\sum_{k=1}^{n}\frac{h_0(k(a_i-x))}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}.$$

From this we have the uniform estimate for $x\in\mathbb{R}$:

$$\left|R_{n,N}^{2.1.1}(f,x)\right|\leqslant\frac{c}{N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{i=0}^{m-1}\left|f'(a_i-0)-f'(a_i+0)\right|\left|\sum_{k=1}^{n}\frac{h_0(k(a_i-x))}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\right|\leqslant\frac{nc(f)}{N^2}.$$

Using Lemma 6 and assuming $\alpha_k = \dfrac{1}{\left(1 - \left(\frac{k}{\mu N}\right)^2\right)^2}$ we have

$$\left| \sum_{k=1}^{n} \frac{h_p(k(a_i - x))}{\left(1 - \left(\frac{k}{\mu N}\right)^2\right)^2} \right| \leqslant \frac{1}{\left|\sin \frac{a_i - x}{2}\right|} \left(\frac{2}{\left(1 - \left(\frac{n}{\mu N}\right)^2\right)^2} + \frac{1}{\left(1 - \left(\frac{1}{\mu N}\right)^2\right)^2} \right) \leqslant \frac{c}{\left|\sin \frac{a_i - x}{2}\right|}.$$

Now we can write

$$\left| R_{n,N}^{2.1.1}(f, x) \right| \leqslant \frac{c}{N^2} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \sum_{i=0}^{m-1} \frac{\left|f'(a_i - 0) - f'(a_i + 0)\right|}{\left|\sin \frac{a_i - x}{2}\right|} \leqslant \frac{c(f, \varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

In the similar way we can get

$$\left| R_{n,N}^{2.1.3}(f, x) \right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R} \text{ and } \left| R_{n,N}^{2.1.3}(f, x) \right| \leqslant \frac{c(f, \varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

Finally, for $R_{n,N}^{2.1}(f, x)$ we can write

$$\left| R_{n,N}^{2.1}(f, x) \right| \leqslant \sum_{i=1}^{4} \left| R_{n,N}^{2.1.i}(f, x) \right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \quad \left| R_{n,N}^{2.1}(f, x) \right| \leqslant \frac{c(f, \varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

Using the same approach we can show that the value $\left| R_{n,N}^{2.2}(f, x) \right|$ has the same estimate as $\left| R_{n,N}^{2.1}(f, x) \right|$, which leads us to (18) and (19). $\qquad\square$

From the previous lemmas and inequality $|R_{n,N}(f, x)| \leqslant \left| R_{n,N}^{1}(f, x) \right| + \left| R_{n,N}^{2}(f, x) \right|$ follow estimates for $|R_{n,N}(f, x)|$:

$$|R_{n,N}(f, x)| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \tag{20}$$

$$|R_{n,N}(f, x)| \leqslant \frac{c(f, \varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{21}$$

3. THE ESTIMATE FOR $\left| a_n^{(2n)}(f) \right|$

Lemma 9. *For the value $a_n^{(2n)}(f)$, where $f \in C_\Omega^2$ and $2n = N$, the estimate $\left| a_n^{(2n)}(f) \right| \leqslant c(f)/N^2$ holds.*

Proof. For each $f \in C_\Omega^2$ the sum $S = \sum_{k=0}^{2n-1} (f(t_k) - f(t_{k+1})) = 0$. We can represent the above sum as $S = S_1 + S_2$, where $S_1 = \sum_{k=0}^{n-1} (f(t_{2k}) - f(t_{2k+1}))$ and $S_2 = \sum_{k=0}^{n-1} (f(t_{2k+1}) - f(t_{2k+2}))$. We can see that $S_1 = -S_2$ and $|S_1| = |S_1 - S_2|/2$. From the above formulas we can write the equation for $S_1 - S_2$:

$$S_1 - S_2 = \sum_{k=0}^{n-1} (f(t_{2k}) - 2f(t_{2k+1}) + f(t_{2k+2})) = \sum_{k=0}^{n-1} \Delta^2 f(t_{2k}). \tag{22}$$

Denote by G the set of numbers $\bigcup_{i=0}^{m} \left\{ k : 0 \leqslant k < n, \ |t_{2k+1} - a_i| \leqslant \frac{2\pi}{N} \right\}$ and $\hat{G} = \{k\}_{k=0}^{n-1} \setminus G$. Rewrite (22) by dividing it into two sums:

$$S_1 - S_2 = \sum_{k \in G} \Delta^2 f(t_{2k}) + \sum_{k \in \hat{G}} \Delta^2 f(t_{2k}).$$

For every $k \in \hat{G}$ we have $|t_{2k+1} - a_i| > 2\pi/N$ $(0 \leqslant i \leqslant m)$ so the points t_{2k}, t_{2k+1}, t_{2k+2} are inside some interval (a_i, a_{i+1}) and the function $f \in C_\Omega^2$ has an absolutely continuous derivative f'' on (a_i, a_{i+1}), therefore we can write $|\Delta^2 f(t_{2k})| \leqslant \left(\frac{2\pi}{N}\right)^2 \max_{x \in [-\pi, \pi]} |f''(x)|$. For $k \in G$ we can write $|\Delta^2 f(t_{2k})| \leqslant c(f)/N$, also note that $|G| \leqslant 2m$. Therefore, we have

$$|S_1| = \frac{|S_1 - S_2|}{2} \leqslant \frac{c(f)}{N}. \tag{23}$$

From (9)

$$a_n^{(2n)}(f) = \frac{1}{N} \sum_{k=0}^{2n-1} f(t_k) \cos \pi k = \frac{1}{N} \sum_{k=0}^{n-1} (f(t_{2k}) - f(t_{2k+1})) = \frac{1}{N} S_1. \tag{24}$$

From (23) and (24) follows $\left| a_n^{(N)}(f) \right| \leqslant c(f)/N^2$. $\qquad\square$

4. THE PROOF OF THEOREM 1

The proof of Theorem 1 consists of two parts: first we prove that the inequalities (3) and (4) of the theorem hold, then we prove that these estimates cannot be improved for all $f \in C_\Omega^2$.

From the inequalities (10), (11), the estimates (13), (14), (20), (21) and Lemma 9 we can easily get (3) and (4). To prove that the order of these estimates cannot be improved we consider the aforementioned 2π-periodic function $f(x) = |x|$, $x \in [-\pi, \pi]$. Obviously, $f \in C_\Omega^2$. Consider only the case when $n < N/2$. From (5) follows the inequality $|f(x) - L_{n,N}(f, x)| \geqslant |f(x) - S_n(f, x)| - |R_{n,N}(f, x)|$. From (20) follows $|R_{n,N}(f, x)| \leqslant c(f)/N$. Therefore, for every $\varepsilon > 0$ we can find a natural number N_0, such that for every $N > N_0$ follows $|R_{n,N}(f, x)| < \varepsilon$. Let $N_0(n)$ be a natural number such that for every $N > N_0(n)$

$$\max_{\substack{x \in E \\ N > N_0(n)}} |R_{n,N}(f, x)| \leqslant \frac{1}{2} \max_{x \in E} |f(x) - S_n(f, x)|,$$

where $E \subset \mathbb{R}$. So, we can write

$$\max_{\substack{x \in E \\ N > N_0(n)}} |f(x) - L_{n,N}(f, x)| \geqslant \frac{1}{2} \max_{x \in E} |f(x) - S_n(f, x)|. \tag{25}$$

Lemma 10. *The following inequalities hold:*

$$\max_{x \in \mathbb{R}} |f(x) - S_n(f, x)| \geqslant c(f)/n, \quad x \in \mathbb{R},$$

$$\max_{|\pi k - x| \geqslant \varepsilon} |f(x) - S_n(f, x)| \geqslant c(f, \varepsilon)/n^2, \quad |\pi k - x| \geqslant \varepsilon.$$

Proof. From [14, p. 443] we have the following representation:

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}, \quad x \in [-\pi, \pi].$$

From the previous equation we can get $f(x) - S_n(f,x) = -\frac{4}{\pi}\sum_{k=n+1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$. For $x = 0$ we have

$$|R_n(f,0)| = \frac{4}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^2} \geqslant c/n.$$

Now we consider the case when $x = \pi/4$ and $n + 1 = 4l$, $l \in \mathbb{N}$. It is easy to show that

$$R_n\left(f, \frac{\pi}{4}\right) = -\frac{2\sqrt{2}}{\pi}\sum_{k=l}^{\infty}\left(\frac{1}{(8k-1)^2} + \frac{1}{(8k+1)^2} - \frac{1}{(8k+3)^2} - \frac{1}{(8k+5)^2}\right) =$$
$$= -\frac{16\sqrt{2}}{\pi}\sum_{k=l}^{\infty}\left(\frac{8k+1}{(8k-1)^2(8k+3)^2} + \frac{8k+3}{(8k+1)^2(8k+5)^2}\right).$$

Hence we have $\left|R_n\left(f, \frac{\pi}{4}\right)\right| \geqslant c/n^2$. $\qquad\square$

From (25) and the above lemma follows

$$\max_{\substack{x\in\mathbb{R}\\N>N_0(n)}}|f(x) - L_{n,N}(f,x)| \geqslant \frac{c}{n}, \quad x\in\mathbb{R},$$

$$\max_{\substack{|\pi k-x|\geqslant\varepsilon\\N>N_0(n)}}|f(x) - L_{n,N}(f,x)| \geqslant \frac{c(\varepsilon)}{n^2}, \quad |x-\pi k|\geqslant\varepsilon.$$

Theorem 1 is proved.

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Приближение непрерывных 2π-периодических кусочно-гладких функций дискретными суммами Фурье

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Пусть $N \geqslant 2$ — некоторое натуральное число. Выберем на вещественной оси N равномерно расположенных точек $t_k = 2\pi k/N + u$ $(0 \leqslant k \leqslant N - 1)$. Обозначим через $L_{n,N}(f) = L_{n,N}(f, x)$ $(1 \leqslant n \leqslant N/2)$ тригонометрический полином порядка n, обладающий наименьшим квадратичным отклонением от f относительно системы $\{t_k\}_{k=0}^{N-1}$. Выберем $m + 1$ точку $-\pi = a_0 < a_1 < \ldots < a_{m-1} < a_m = \pi$, где $m \geqslant 2$, и обозначим $\Omega = \{a_i\}_{i=0}^m$. Через C_Ω^r обозначим класс 2π-периодических непрерывных функций f, r-раз дифференцируемых на каждом сегменте $\Delta_i = [a_i, a_{i+1}]$, причем производная $f^{(r)}$ на каждом Δ_i абсолютно непрерывна. В данной работе рассмотрена задача приближения функций $f \in C_\Omega^2$ полиномами $L_{n,N}(f, x)$. Показано, что вместо оценки $|f(x) - L_{n,N}(f, x)| \leqslant c \ln n/n$, которая следует из известного неравенства Лебега, найдена точная по порядку оценка $|f(x) - L_{n,N}(f, x)| \leqslant c/n$ $(x \in \mathbb{R})$, которая равномерна относительно n $(1 \leqslant n \leqslant N/2)$. Кроме того, найдена локальная оценка $|f(x) - L_{n,N}(f, x)| \leqslant c(\varepsilon)/n^2$ $(|x - a_i| \geqslant \varepsilon)$, которая также равномерна относительно n $(1 \leqslant n \leqslant N/2)$. Доказательства этих оценок основаны на сравнении дискретных и непрерывных конечных сумм ряда Фурье.

Ключевые слова: приближение функций, тригонометрические полиномы, ряд Фурье.

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