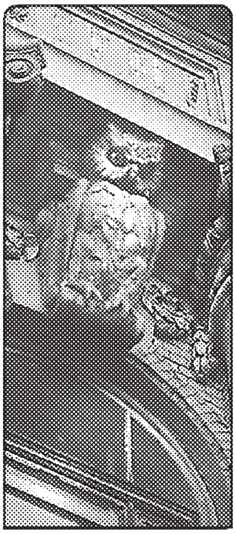

НАУЧНЫЙ ОТДЕЛ

МАТЕМАТИКА

Approximation of Continuous 2π-Periodic Piecewise Smooth Functions by Discrete Fourier Sums

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Let N be a natural number greater than 1. Select N uniformly distributed points $t_k = 2\pi k/N + u$ $(0 \leqslant k \leqslant N-1)$, and denote by $L_{n,N}(f) = L_{n,N}(f,x)$ $(1 \leqslant n \leqslant N/2)$ the trigonometric polynomial of order n possessing the least quadratic deviation from f with respect to the system $\{t_k\}_{k=0}^{N-1}$. Select $m+1$ points $-\pi = a_0 < a_1 < \ldots < a_{m-1} < a_m = \pi$, where $m \geqslant 2$, and denote $\Omega = \{a_i\}_{i=0}^{m}$. Denote by C_Ω^r a class of 2π-periodic continuous functions f, where f is r-times differentiable on each segment $\Delta_i = [a_i, a_{i+1}]$ and $f^{(r)}$ is absolutely continuous on Δ_i. In the present article we consider the problem of approximation of functions $f \in C_\Omega^2$ by the polynomials $L_{n,N}(f,x)$. We show that instead of the estimate $|f(x) - L_{n,N}(f,x)| \leqslant c \ln n/n$, which follows from the well-known Lebesgue inequality, we found an exact order estimate $|f(x) - L_{n,N}(f,x)| \leqslant c/n$ $(x \in \mathbb{R})$ which is uniform with respect to n $(1 \leqslant n \leqslant N/2)$. Moreover, we found a local estimate $|f(x) - L_{n,N}(f,x)| \leqslant c(\varepsilon)/n^2$ $(|x - a_i| \geqslant \varepsilon)$ which is also uniform with respect to n $(1 \leqslant n \leqslant N/2)$. The proofs of these estimations are based on comparing of approximating properties of discrete and continuous finite Fourier series.

Keywords: function approximation, trigonometric polynomials, Fourier series.

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INTRODUCTION

We begin by establishing some notations. Let Ω be a set of $m+1$ points $\{a_i\}_{i=0}^{m}$ $(m > 2)$ such that $-\pi = a_0 < a_1 < \ldots < a_{m-1} < a_m = \pi$. We denote by $C_\Omega^{0,r}$ the class of 2π-periodic functions with r absolutely continuous derivatives on each interval (a_i, a_{i+1}) and by C_Ω^r the subclass of all continuous functions in $C_\Omega^{0,r}$ (here we say that a function f is absolutely continuous on an interval (a,b) if the function $\overline{f}$ is absolutely continuous on the segment $[a,b]$, where $\overline{f}(x)=f(x)$ for $x\in(a,b)$, $\overline{f}(a)=f(a+0)$, and $\overline{f}(b)=f(b-0)$).

We denote by $L_{n,N}(f,x)$ a trigonometric polynomial of order n possessing the least quadratic deviation from the function f at the points $\{t_j\}_{i=0}^{N-1}$, where $t_j=u+2\pi j/N$, $n\leqslant N/2$, $N\geqslant 2$, and $u\in\mathbb{R}$. In other words, $L_{n,N}(f,x)$ provides the minimum for the sum $\sum_{j=0}^{N-1}|f(t_j)-T_n(t_j)|^2$ on the set of all trigonometric polynomials of order at most n. To read more about function approximation by trigonometric polynomials see [1–10].

Also, in this paper we denote by c or $c(b_1,b_2,\ldots,b_k)$ some positive constants, which depend only on specified arguments (if any) and may vary from line to line, and by $S_n(f,x)$ the n-th partial sum of the Fourier series of the function f. We also note that it is easy to show that the Fourier series of any $f\in C_\Omega^2$ converges uniformly on $\mathbb{R}$ and the following representation is possible:

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right), \tag{1}$$

where

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos ktdt,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ktdt. \tag{2}$$

The goal of this work is to estimate the value $|f(x)-L_{n,N}(f,x)|$ for $f\in C_\Omega^2$. Note that the special case of this problem is considered in [11], where the value $|f(x)-L_{n,N}(f,x)|$ is estimated for a 2π-periodic function $f(x)=|x|$ ($x\in[-\pi,\pi]$). In this work, we generalize the results from [11] for any function $f\in C_\Omega^2$, as stated in the following theorem:

Theorem 1. *For $f\in C_\Omega^2$ the following inequalities hold:*

$$|f(x)-L_{n,N}(f,x)|\leqslant\frac{c(f)}{n},\quad x\in\mathbb{R}, \tag{3}$$

$$|f(x)-L_{n,N}(f,x)|\leqslant\frac{c(f,\varepsilon)}{n^2},\quad x\in|x-a_i|\geqslant\varepsilon. \tag{4}$$

The order of these estimates cannot be improved.

To prove this theorem we use a lemma from [12]:

Lemma 1 (Sharapudinov, [12]). *If the Fourier series of f converges at the points $t_k=u+2k\pi/N$, then the representation*

$$L_{n,N}(f,x)=S_n(f,x)+R_{n,N}(f,x), \tag{5}$$

where

$$R_{n,N}(f,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi}D_n(x-t)\cos\mu N(u-t)f(t)dt, \tag{6}$$

holds true, where $2n<N$ and $D_n(x)$ is the Dirichlet kernel:

$$D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx. \tag{7}$$

This lemma considers only the case $2n < N$. If $2n = N$ (when N is even) we can write (see [12])

$$L_{n,2n}(f,x) = L_{n-1,2n}(f,x) + a_n^{(2n)}(f)\cos n(x-u), \tag{8}$$

where

$$a_n^{(2n)}(f) = \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k)\cos n(t_k - u). \tag{9}$$

To prove the inequalities (3) and (4) from Theorem 1 we use the formulas

$$|f(x) - L_{n,N}(f,x)| \leqslant |f(x) - S_n(f,x)| + |R_{n,N}(f,x)|, \quad n < N/2, \tag{10}$$

$$|f(x) - L_{n,N}(f,x)| \leqslant |f(x) - S_{n-1}(f,x)| + |R_{n-1,N}(f,x)| + \left|a_n^{(2n)}(f)\right|, \quad n = N/2, \tag{11}$$

which immediately follow from (5) and (8).

The estimates for the values $|f(x) - S_n(f,x)|$, $|R_{n,N}(f,x)|$, and $|a_n^{(2n)}(f)|$ are found in the following sections.

1. THE ESTIMATE FOR $|f(x) - S_n(f,x)|$

To estimate the value $|f(x) - S_n(f,x)|$ we need the following lemma.

Lemma 2. *For $f \in C^2_\Omega$ the following inequality holds:*

$$\left|\int_{-\pi}^{\pi} f(t)h_p(k(t+\alpha))dt\right| \leqslant \frac{c(f)}{k^2},$$

where $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and

$$h_p(x) = \begin{cases} \cos x, & p = 0, \\ \sin x, & p = 1. \end{cases} \tag{12}$$

Proof. Performing integration by parts two times we have

$$\int_{-\pi}^{\pi} f(t)h_p(k(t+\alpha))dt = \frac{(-1)^{p+1}}{k}\int_{-\pi}^{\pi} f'(t)h_{1-p}(k(t+\alpha))dt =$$

$$= \frac{1}{k^2}\left[\sum_{i=0}^{m-1}\left(f'(a_i - 0) - f'(a_i + 0)\right)h_p(k(a_i+\alpha)) - \int_{-\pi}^{\pi} f''(t)h_p(k(t+\alpha))dt\right].$$

From this we can get the estimate

$$\left|\int_{-\pi}^{\pi} f(t)h_p(k(t+\alpha))dt\right| \leqslant \frac{1}{k^2}\left[\sum_{i=0}^{m-1}\left|f'(a_i - 0) - f'(a_i + 0)\right| + \int_{-\pi}^{\pi}\left|f''(t)\right|dt\right] \leqslant \frac{c(f)}{k^2}.$$

□

Lemma 3. *For $f \in C^2_\Omega$ the following inequalities hold:*

$$|f(x) - S_n(f,x)| \leqslant \frac{c(f)}{n}, \quad x \in \mathbb{R}, \tag{13}$$

$$|f(x) - S_n(f,x)| \leqslant \frac{c(f,\varepsilon)}{n^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{14}$$

Proof. Here we prove only (13) because the proof for inequality (14) can be found in [13, Theorem 2.1]. Using (1) and (2) we can write

$$f(x) - S_n(f,x) = \sum_{k=n+1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Applying Lemma 2 to (2) we get $|a_k| \leqslant c(f)/k^2$ and $|b_k| \leqslant c(f)/k^2$, which gives us

$$|f(x) - S_n(f,x)| \leqslant \sum_{k=n+1}^{\infty} (|a_k| + |b_k|) \leqslant c(f)/n.$$

□

2. THE ESTIMATE FOR $|R_{n,N}(f,x)|$

From (6) and (7) follows $R_{n,N}(f,x) = R^1_{n,N}(f,x) + R^2_{n,N}(f,x)$, where

$$R^1_{n,N}(f,x) = \frac{1}{\pi} \sum_{\mu=1}^{\infty} \int_{-\pi}^{\pi} f(t) \cos \mu N(u-t) dt,$$

$$R^2_{n,N}(f,x) = \frac{2}{\pi} \sum_{\mu=1}^{\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \cos k(x-t) \cos \mu N(u-t) dt. \tag{15}$$

Obviously, $|R_{n,N}(f,x)| \leqslant \left|R^1_{n,N}(f,x)\right| + \left|R^2_{n,N}(f,x)\right|$. The values $\left|R^1_{n,N}(f,x)\right|$ and $\left|R^2_{n,N}(f,x)\right|$ are estimated later in this section, but first we prove three auxiliary lemmas.

Lemma 4. *For $f \in C^{0,1}_\Omega$ the following holds:*

$$\int_{-\pi}^{\pi} f(t) h_p(k(t-x)) h_q(\mu N(t-u)) dt =$$

$$= \frac{(-1)^q \mu N}{(\mu N)^2 - k^2} \sum_{i=0}^{m-1} (f(a_i - 0) - f(a_i + 0)) h_p(k(a_i - x)) h_{1-q}(\mu N(a_i - u)) -$$

$$- \frac{(-1)^q \mu N}{(\mu N)^2 - k^2} \int_{-\pi}^{\pi} f'(t) h_p(k(t-x)) h_{1-q}(\mu N(t-u)) dt +$$

$$+ \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2} \sum_{i=0}^{m-1} (f(a_i - 0) - f(a_i + 0)) h_{1-p}(k(a_i - x)) h_q(\mu N(a_i - u)) -$$

$$- \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2} \int_{-\pi}^{\pi} f'(t) h_{1-p}(k(t-x)) h_q(\mu N(t-u)) dt. \tag{16}$$

Proof. Perform integration by parts:

$$\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N}\sum_{i=0}^{m-1}\left(f(a_i-0)-f(a_i+0)\right)h_p(k(a_i-x))h_{1-q}(\mu N(a_i-u))-$$

$$-\frac{(-1)^q}{\mu N}\int_{-\pi}^{\pi} f^{'}(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt+$$

$$+\frac{(-1)^{p+q}k}{\mu N}\int_{-\pi}^{\pi} f(t)h_{1-p}(k(t-x))h_{1-q}(\mu N(t-u))dt. \qquad (17)$$

Repeat integration by parts for the last integral in (17):

$$\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N}\sum_{i=0}^{m-1}\left(f(a_i-0)-f(a_i+0)\right)h_p(k(a_i-x))h_{1-q}(\mu N(a_i-u))-$$

$$-\frac{(-1)^q}{\mu N}\int_{-\pi}^{\pi} f^{'}(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt+$$

$$+\frac{(-1)^{1+p}k}{(\mu N)^2}\sum_{i=0}^{m-1}\left(f(a_i-0)-f(a_i+0)\right)h_{1-p}(k(a_i-x))h_q(\mu N(a_i-u))-$$

$$-\frac{(-1)^{1+p}k}{(\mu N)^2}\int_{-\pi}^{\pi} f^{'}(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt+$$

$$+\frac{k^2}{(\mu N)^2}\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt.$$

By moving the last integral from the right side to the left and dividing both sides by $\frac{(\mu N)^2-k^2}{(\mu N)^2}$ we get (16). □

Corollary 1. *If $f \in C^2_\Omega$, then $f(a_i-0)-f(a_i+0)=0$, so we can write* (16) *as*

$$\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt = \frac{(-1)^q\mu N}{(\mu N)^2-k^2}\int_{-\pi}^{\pi} f^{'}(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt-$$

$$-\frac{(-1)^{1+p}k}{(\mu N)^2-k^2}\int_{-\pi}^{\pi} f^{'}(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt.$$

Lemma 5. *The following estimate holds:*

$$\left|\sum_{k=1}^{n} h_p(kx)\right| \leqslant \frac{1}{\left|\sin\frac{x}{2}\right|}.$$

Proof. The proof is obvious and follows from well-known formulas

$$\sum_{k=1}^{n}\sin(kx) = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}, \quad \sum_{k=1}^{n}\cos(kx) = \frac{\cos\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

□

Lemma 6. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a monotonous sequence (either increasing or decreasing) of n positive numbers. The following holds:*

$$\left|\sum_{k=1}^{n} \alpha_k h_p(kx)\right| \leqslant \frac{2\alpha_n + \alpha_1}{\left|\sin\frac{x}{2}\right|}.$$

Proof. After performing Abel transformation (summation by parts) we have:

$$\sum_{k=1}^{n} \alpha_k h_p(kx) = \alpha_n \sum_{j=1}^{n} h_p(jx) - \sum_{k=1}^{n-1}(\alpha_{k+1} - \alpha_k)\sum_{j=1}^{k} h_p(jx).$$

Using Lemma 5 and the fact that $\sum_{k=1}^{n}|\alpha_{k+1} - \alpha_k| = |\sum_{k=1}^{n}\alpha_{k+1} - \alpha_k|$ we can write

$$\left|\sum_{k=1}^{n} \alpha_k h_p(kx)\right| \leqslant \alpha_n\left|\sum_{j=1}^{n} h_p(jx)\right| + \sum_{k=1}^{n-1}|\alpha_{k+1} - \alpha_k|\left|\sum_{j=1}^{k} h_p(jx)\right| \leqslant$$
$$\leqslant \frac{1}{\left|\sin\frac{x}{2}\right|}\left(\alpha_n + \left|\sum_{k=1}^{n-1}\alpha_{k+1} - \alpha_k\right|\right) \leqslant \frac{2\alpha_n + \alpha_1}{\left|\sin\frac{x}{2}\right|}.$$

□

Lemma 7. *The following inequality holds:* $\left|R^1_{n,N}(f,x)\right| \leqslant c(f)/N^2$.

Proof. Using Lemma 2 we have

$$\left|R^1_{n,N}(f,x)\right| \leqslant \frac{1}{\pi}\sum_{\mu=1}^{\infty}\left|\int_{-\pi}^{\pi} f(t)\cos\mu N(u-t)dt\right| \leqslant \frac{c(f)}{N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2} \leqslant \frac{c(f)}{N^2}.$$

□

Lemma 8. *The following estimates hold:*

$$\left|R^2_{n,N}(f,x)\right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \tag{18}$$

$$\left|R^2_{n,N}(f,x)\right| \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{19}$$

Proof. Rewrite (15) using (12):

$$R^2_{n,N}(f,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\sum_{k=1}^{n}\int_{-\pi}^{\pi}f(t)h_0(k(t-x))h_0(\mu N(t-u))dt.$$

Using Corollary 1 we rewrite the above formula as follows:

$$R^2_{n,N}(f,x)=\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu}\sum_{k=1}^{n}\frac{1}{1-\left(\frac{k}{\mu N}\right)^2}\int_{-\pi}^{\pi}f'(t)h_0(k(t-x))h_1(\mu N(t-u))dt+$$
$$+\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{k=1}^{n}\frac{k}{N}\frac{1}{1-\left(\frac{k}{\mu N}\right)^2}\int_{-\pi}^{\pi}f'(t)h_1(k(t-x))h_0(\mu N(t-u))dt=$$
$$=R^{2.1}_{n,N}(f,x)+R^{2.2}_{n,N}(f,x).$$

For brevity we only consider here estimation of $\left|R^{2.1}_{n,N}(f,x)\right|$ because $\left|R^{2.2}_{n,N}(f,x)\right|$ can be estimated in almost the same way. Obviously, $f'\in C^{0,1}_{\Omega}$, so we can apply Lemma 4 to $R^{2.1}_{n,N}(f,x)$:

$$R^{2.1}_{n,N}(f,x)=\frac{-2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\sum_{i=0}^{m-1}\Big(f'(a_i-0)-f'(a_i+0)\Big)\times$$
$$\times h_0(k(a_i-x))h_0(\mu N(a_i-u))+$$
$$+\frac{2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\int_{-\pi}^{\pi}f''(t)h_0(k(t-x))h_0(\mu N(t-u))dt+$$
$$+\frac{-2}{\pi N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\sum_{i=0}^{m-1}\Big(f'(a_i-0)-f'(a_i+0)\Big)\times$$
$$\times h_1(k(a_i-x))h_1(\mu N(a_i-u))+$$
$$+\frac{2}{\pi N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\int_{-\pi}^{\pi}f''(t)h_1(k(t-x))h_1(\mu N(t-u))dt=$$
$$=R^{2.1.1}_{n,N}(f,x)+R^{2.1.2}_{n,N}(f,x)+R^{2.1.3}_{n,N}(f,x)+R^{2.1.4}_{n,N}(f,x).$$

Begin with $R^{2.1.2}_{n,N}(f,x)$. Applying Lemma 4 we get

$$R^{2.1.2}_{n,N}(f,x)=\frac{2}{\pi N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\Big(f''(a_i-0)-f''(a_i+0)\Big)\times$$
$$\times h_0(k(a_i-x))h_1(\mu N(a_i-u))+$$

$$+\frac{-2}{\pi N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int\limits_{-\pi}^{\pi} f'''(t)h_0(k(t-x))h_1(\mu N(t-u))dt+$$

$$+\frac{-2}{\pi N^4}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left(f''(a_i-0)-f''(a_i+0)\right)\times$$

$$\times h_1(k(a_i-x))h_0(\mu N(a_i-u))+$$

$$+\frac{2}{\pi N^4}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int\limits_{-\pi}^{\pi} f'''(t)h_1(k(t-x))h_0(\mu N(t-u))dt.$$

From this we can get the estimate

$$\left|R_{n,N}^{2.1.2}(f,x)\right| \leqslant \frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left|f''(a_i-0)-f''(a_i+0)\right|+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\sum_{k=1}^{n}\frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int\limits_{-\pi}^{\pi}\left|f'''(t)\right|dt+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k/N}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\sum_{i=0}^{m-1}\left|f''(a_i-0)-f''(a_i+0)\right|+$$

$$+\frac{c}{N^3}\sum_{\mu=1}^{\infty}\frac{1}{\mu^4}\sum_{k=1}^{n}\frac{k/N}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^3}\int\limits_{-\pi}^{\pi}\left|f'''(t)\right|dt \leqslant \frac{c(f)}{N^2}.$$

In the same way we can get $\left|R_{n,N}^{2.1.4}(f,x)\right| \leqslant c(f)/N^2$. Now we consider $\left|R_{n,N}^{2.1.1}(f,x)\right|$ and $\left|R_{n,N}^{2.1.3}(f,x)\right|$. We will estimate here only $\left|R_{n,N}^{2.1.1}(f,x)\right|$ because the other one can be estimated in the similar way. After a simple transformation we have

$$R_{n,N}^{2.1.1}(f,x)=\frac{-2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{i=0}^{m-1}\left(f'(a_i-0)-f'(a_i+0)\right)h_0(\mu N(a_i-u))\sum_{k=1}^{n}\frac{h_0(k(a_i-x))}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}.$$

From this we have the uniform estimate for $x\in\mathbb{R}$:

$$\left|R_{n,N}^{2.1.1}(f,x)\right| \leqslant \frac{c}{N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{i=0}^{m-1}\left|f'(a_i-0)-f'(a_i+0)\right|\left|\sum_{k=1}^{n}\frac{h_0(k(a_i-x))}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\right| \leqslant \frac{nc(f)}{N^2}.$$

Using Lemma 6 and assuming $\alpha_k = \frac{1}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}$ we have

$$\left|\sum_{k=1}^{n} \frac{h_p(k(a_i - x))}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)^2}\right| \leqslant \frac{1}{\left|\sin\frac{a_i - x}{2}\right|}\left(\frac{2}{\left(1-\left(\frac{n}{\mu N}\right)^2\right)^2} + \frac{1}{\left(1-\left(\frac{1}{\mu N}\right)^2\right)^2}\right) \leqslant \frac{c}{\left|\sin\frac{a_i - x}{2}\right|}.$$

Now we can write

$$\left|R_{n,N}^{2.1.1}(f,x)\right| \leqslant \frac{c}{N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{i=0}^{m-1}\frac{\left|f'(a_i-0) - f'(a_i+0)\right|}{\left|\sin\frac{a_i - x}{2}\right|} \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

In the similar way we can get

$$\left|R_{n,N}^{2.1.3}(f,x)\right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R} \text{ and } \left|R_{n,N}^{2.1.3}(f,x)\right| \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

Finally, for $R_{n,N}^{2.1}(f,x)$ we can write

$$\left|R_{n,N}^{2.1}(f,x)\right| \leqslant \sum_{i=1}^{4}\left|R_{n,N}^{2.1.i}(f,x)\right| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \quad \left|R_{n,N}^{2.1}(f,x)\right| \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon.$$

Using the same approach we can show that the value $\left|R_{n,N}^{2.2}(f,x)\right|$ has the same estimate as $\left|R_{n,N}^{2.1}(f,x)\right|$, which leads us to (18) and (19). ☐

From the previous lemmas and inequality $|R_{n,N}(f,x)| \leqslant \left|R_{n,N}^{1}(f,x)\right| + \left|R_{n,N}^{2}(f,x)\right|$ follow estimates for $|R_{n,N}(f,x)|$:

$$|R_{n,N}(f,x)| \leqslant \frac{nc(f)}{N^2}, \quad x \in \mathbb{R}, \tag{20}$$

$$|R_{n,N}(f,x)| \leqslant \frac{c(f,\varepsilon)}{N^2}, \quad |x - a_i| \geqslant \varepsilon. \tag{21}$$

3. THE ESTIMATE FOR $\left|a_n^{(2n)}(f)\right|$

Lemma 9. *For the value $a_n^{(2n)}(f)$, where $f \in C_\Omega^2$ and $2n = N$, the estimate $\left|a_n^{(2n)}(f)\right| \leqslant c(f)/N^2$ holds.*

Proof. For each $f \in C_\Omega^2$ the sum $S = \sum_{k=0}^{2n-1}\left(f(t_k) - f(t_{k+1})\right) = 0$. We can represent the above sum as $S = S_1 + S_2$, where $S_1 = \sum_{k=0}^{n-1}\left(f(t_{2k}) - f(t_{2k+1})\right)$ and $S_2 = \sum_{k=0}^{n-1}\left(f(t_{2k+1}) - f(t_{2k+2})\right)$. We can see that $S_1 = -S_2$ and $|S_1| = |S_1 - S_2|/2$. From the above formulas we can write the equation for $S_1 - S_2$:

$$S_1 - S_2 = \sum_{k=0}^{n-1}\left(f(t_{2k}) - 2f(t_{2k+1}) + f(t_{2k+2})\right) = \sum_{k=0}^{n-1}\Delta^2 f\left(t_{2k}\right). \tag{22}$$

Denote by G the set of numbers $\bigcup_{i=0}^{m}\left\{k : 0 \leqslant k < n,\ |t_{2k+1} - a_i| \leqslant \frac{2\pi}{N}\right\}$ and $\hat{G} = \{k\}_{k=0}^{n-1} \setminus G$. Rewrite (22) by dividing it into two sums:

$$S_1 - S_2 = \sum_{k\in G}\Delta^2 f(t_{2k}) + \sum_{k\in\hat{G}}\Delta^2 f(t_{2k}).$$

For every $k \in \hat{G}$ we have $|t_{2k+1} - a_i| > 2\pi/N$ $(0 \leqslant i \leqslant m)$ so the points t_{2k}, t_{2k+1}, t_{2k+2} are inside some interval (a_i, a_{i+1}) and the function $f \in C^2_\Omega$ has an absolutely continuous derivative f'' on (a_i, a_{i+1}), therefore we can write $|\Delta^2 f(t_{2k})| \leqslant \left(\frac{2\pi}{N}\right)^2 \max_{x\in[-\pi,\pi]} \left|f''(x)\right|$. For $k \in G$ we can write $|\Delta^2 f(t_{2k})| \leqslant c(f)/N$, also note that $|G| \leqslant 2m$. Therefore, we have

$$|S_1| = \frac{|S_1 - S_2|}{2} \leqslant \frac{c(f)}{N}. \tag{23}$$

From (9)

$$a_n^{(2n)}(f) = \frac{1}{N}\sum_{k=0}^{2n-1} f(t_k)\cos\pi k = \frac{1}{N}\sum_{k=0}^{n-1}\left(f(t_{2k}) - f(t_{2k+1})\right) = \frac{1}{N}S_1. \tag{24}$$

From (23) and (24) follows $\left|a_n^{(N)}(f)\right| \leqslant c(f)/N^2$. □

4. THE PROOF OF THEOREM 1

The proof of Theorem 1 consists of two parts: first we prove that the inequalities (3) and (4) of the theorem hold, then we prove that these estimates cannot be improved for all $f \in C^2_\Omega$.

From the inequalities (10), (11), the estimates (13), (14), (20), (21) and Lemma 9 we can easily get (3) and (4). To prove that the order of these estimates cannot be improved we consider the aforementioned 2π-periodic function $f(x) = |x|$, $x \in [-\pi, \pi]$. Obviously, $f \in C^2_\Omega$. Consider only the case when $n < N/2$. From (5) follows the inequality $|f(x) - L_{n,N}(f,x)| \geqslant |f(x) - S_n(f,x)| - |R_{n,N}(f,x)|$. From (20) follows $|R_{n,N}(f,x)| \leqslant c(f)/N$. Therefore, for every $\varepsilon > 0$ we can find a natural number N_0, such that for every $N > N_0$ follows $|R_{n,N}(f,x)| < \varepsilon$. Let $N_0(n)$ be a natural number such that for every $N > N_0(n)$

$$\max_{\substack{x\in E\\ N>N_0(n)}} |R_{n,N}(f,x)| \leqslant \frac{1}{2}\max_{x\in E}|f(x) - S_n(f,x)|,$$

where $E \subset \mathbb{R}$. So, we can write

$$\max_{\substack{x\in E\\ N>N_0(n)}} |f(x) - L_{n,N}(f,x)| \geqslant \frac{1}{2}\max_{x\in E}|f(x) - S_n(f,x)|. \tag{25}$$

Lemma 10. *The following inequalities hold:*

$$\max_{x\in\mathbb{R}}|f(x) - S_n(f,x)| \geqslant c(f)/n, \quad x \in \mathbb{R},$$
$$\max_{|\pi k - x|\geqslant\varepsilon}|f(x) - S_n(f,x)| \geqslant c(f,\varepsilon)/n^2, \quad |\pi k - x| \geqslant \varepsilon.$$

Proof. From [14, p. 443] we have the following representation:

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}, \quad x \in [-\pi, \pi].$$

From the previous equation we can get $f(x) - S_n(f,x) = -\frac{4}{\pi}\sum_{k=n+1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$. For $x = 0$ we have

$$|R_n(f,0)| = \frac{4}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^2} \geqslant c/n.$$

Now we consider the case when $x = \pi/4$ and $n+1 = 4l$, $l \in \mathbb{N}$. It is easy to show that

$$R_n\left(f,\frac{\pi}{4}\right) = -\frac{2\sqrt{2}}{\pi}\sum_{k=l}^{\infty}\left(\frac{1}{(8k-1)^2} + \frac{1}{(8k+1)^2} - \frac{1}{(8k+3)^2} - \frac{1}{(8k+5)^2}\right) =$$

$$= -\frac{16\sqrt{2}}{\pi}\sum_{k=l}^{\infty}\left(\frac{8k+1}{(8k-1)^2(8k+3)^2} + \frac{8k+3}{(8k+1)^2(8k+5)^2}\right).$$

Hence we have $\left|R_n\left(f,\frac{\pi}{4}\right)\right| \geqslant c/n^2$. □

From (25) and the above lemma follows

$$\max_{\substack{x\in\mathbb{R}\\ N>N_0(n)}} |f(x) - L_{n,N}(f,x)| \geqslant \frac{c}{n}, \quad x \in \mathbb{R},$$

$$\max_{\substack{|\pi k - x|\geqslant\varepsilon\\ N>N_0(n)}} |f(x) - L_{n,N}(f,x)| \geqslant \frac{c(\varepsilon)}{n^2}, \quad |x - \pi k| \geqslant \varepsilon.$$

Theorem 1 is proved.

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Приближение непрерывных 2π-периодических кусочно-гладких функций дискретными суммами Фурье

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Пусть $N \geqslant 2$ — некоторое натуральное число. Выберем на вещественной оси N равномерно расположенных точек $t_k = 2\pi k/N + u$ $(0 \leqslant k \leqslant N-1)$. Обозначим через $L_{n,N}(f) = L_{n,N}(f,x)$ $(1 \leqslant n \leqslant N/2)$ тригонометрический полином порядка n, обладающий наименьшим квадратичным отклонением от f относительно системы $\{t_k\}_{k=0}^{N-1}$. Выберем $m+1$ точку $-\pi = a_0 < a_1 < \ldots < a_{m-1} < a_m = \pi$, где $m \geqslant 2$, и обозначим $\Omega = \{a_i\}_{i=0}^{m}$. Через C_Ω^r обозначим класс 2π-периодических непрерывных функций f, r-раз дифференцируемых на каждом сегменте $\Delta_i = [a_i, a_{i+1}]$, причем производная $f^{(r)}$ на каждом Δ_i абсолютно непрерывна. В данной работе рассмотрена задача приближения функций $f \in C_\Omega^2$ полиномами $L_{n,N}(f,x)$. Показано, что вместо оценки $|f(x) - L_{n,N}(f,x)| \leqslant c \ln n/n$, которая следует из известного неравенства Лебега, найдена точная по порядку оценка $|f(x) - L_{n,N}(f,x)| \leqslant c/n$ $(x \in \mathbb{R})$, которая равномерна относительно n $(1 \leqslant n \leqslant N/2)$. Кроме того, найдена локальная оценка $|f(x) - L_{n,N}(f,x)| \leqslant c(\varepsilon)/n^2$ $(|x - a_i| \geqslant \varepsilon)$, которая также равномерна относительно n $(1 \leqslant n \leqslant N/2)$. Доказательства этих оценок основаны на сравнении дискретных и непрерывных конечных сумм ряда Фурье.

Ключевые слова: приближение функций, тригонометрические полиномы, ряд Фурье.

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