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The Il'in Spectral Method for Determination of the Properties of the Basis Property and the Uniform Convergence of Biorthogonal Expansions on a Finite Interval

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The paper discusses the basics of the spectral method of V. A. Il'in on an example of a simple second order differential operator on a segment of the number line. The first theorem of Il'in on the unconditional basis property is stated. Its detailed proof is given. A chain of generalizations of this theorem is traced. A recently established a theorem on the unconditional basis property for the differential operators with general integral boundary conditions is formulated. The substantiation of the statements about the uniform convergence of biorthogonal expansions of functions using the Il'in method is presented. The main theorems, including, the recently established theorem for operators with integral boundary conditions are formulated.

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