Pieri Formulae and Specialisation of Super Jacobi Polynomials

A. N. Sergeev, E. D. Zharinov

Alexander N. Sergeev, https://orcid.org/0000-0002-9637-0705, Saratov State University, 83 Astrakhanskaya St., Saratov 410012, Russia, SergeevAN@info.sgu.ru, asergeev@hse.ru

Egor D. Zharinov, Saratov State University, 83 Astrakhanskaya St., Saratov 410012, Russia, egor.zharinov@gmail.com

We give a new proof of the fact that the Euler supercharacters of the Lie superalgebra $\mathfrak{osp}(2m+1,2n)$ can be obtained as a certain limit of the super Jacobi polynomials. The known proof was not direct one and it was mostly based on calculations. In this paper we propose more simple and more conceptional proof. The main idea is to use the Pieri formulae from the beginning. It turns out that the super Jacobi polynomials and their specialisations can be uniquely characterised by two properties. The first one is that they are eigenfunctions of CMS operator and the second one is that they satisfy the Pieri formulae. As by product we get some interesting identities involving a Young diagram and rational functions. We hope that our approach can be useful in many similar cases.

Keywords: quantum CMS operator, Pieri formula, super Jacobi polynomial, superalgebra, Euler supercharacter.

INTRODUCTION

Generalisations and specialisations of Macdonald polynomials play an important role in many fields of mathematics (see for example [1–3] and [4–6]). Super Jacobi polynomials are one of such generalisations (see [7]). They can be naturally connected to the theory of Lie superalgebras and deformed quantum Calogero – Moser – Sutherland (CMS) systems. Let J_λ be the family of super Jacobi polynomials in $m + n$ indeterminates (see [7]) labeled by the set of partitions λ such that $\lambda_{m+1} \leqslant n$. It has been proved in the paper [8] that

$$\lim_{(p,q)\to(-1,0)} \lim_{k\to-1} J_\lambda = (-1)^{s(\lambda)} E_\lambda,$$

where E_λ is the Euler supercharacter (with a special choice of the parabolic subalgebra) of the Lie superalgebra $\mathfrak{osp}(2m+1,2n)$ and p, q, k are the parameters of the super Jacobi polynomial and the number $s(\lambda)$ is defined in Theorem 6.

This result is interesting and important from the representation theory point of view and from the point of view of CMS systems as well. But the proof given in the paper [8] is rather involved. It uses the infinite dimensional version of the Jacobi symmetric polynomials, Okounkov formula for Jacobi symmetric functions, generalisation of the Weyl type formula for super Schur polynomials and some other more technical means.

The goal of this paper is to give more simple and more conceptual proof of that result. In the present paper we use two main properties of super Jacobi polynomials. The first one is that they are eigenfunctions of the deformed Calogero – Moser – Sutherland

operator and the second property is that they satisfy the Pieri identity. This two properties define them uniquely and the same is true for the Euler supercharacters as well. So instead of calculating the limit of the super Jacobi polynomials we calculate the limit of the CMS operator (which is trivial) and the limit of the coefficients of the Pieri formulae. The last limit can be calculated using some simple identities with rational functions and Young diagrams. On the other side we have an explicit formula for the Euler supercharacters obtained by V. Serganova [9]. Using this explicit formula we prove the corresponding Pieri formula and the fact that the Euler supercharacters are eigenfunctions of the limiting CMS operator. As a result we get the coincidence of the family of specialised super Jacobi polynomials and the Euler supercharacters. We believe that this method can be applied in many similar cases. For example using this approach it is easy to prove the Weyl type formula for super Schur polynomials.

1. SUPER JACOBI POLYNOMIALS

In this section we define super Jacobi polynomials using the fact that they satisfy the Pieri formula and that they are eigenfunctions of the deformed Calogero – Moser – Sutherland operator. The deformed CMS operator of type $BC_{m,n}$ has the following form (see [7, p. 1712])

$$\mathscr{L} = \sum_{i=1}^{m} \partial_{x_i}^2 + k \sum_{j=1}^{n} \partial_{y_j}^2 - k \sum_{i<j}^{m} \left(\frac{x_i + x_j}{x_i - x_j}(\partial_{x_i} - \partial_{x_j}) + \frac{x_i x_j + 1}{x_i x_j - 1}(\partial_{x_i} + \partial_{x_j}) \right) -$$
$$- \sum_{i<j}^{n} \left(\frac{y_i + y_j}{y_i - y_j}(\partial_{y_i} - \partial_{y_j}) + \frac{y_i y_j + 1}{y_i y_j - 1}(\partial_{y_i} + \partial_{y_j}) \right) -$$
$$- \sum_{i=1}^{m} \left(p \frac{x_i + 1}{x_i - 1} + 2q \frac{x_i^2 + 1}{x_i^2 - 1} \right) \partial_{x_i} - k \sum_{j=1}^{n} \left(r \frac{y_j + 1}{y_j - 1} + 2s \frac{y_j^2 + 1}{y_j^2 - 1} \right) \partial_{y_j} -$$
$$- \sum_{i,j} \left(\frac{x_i + y_j}{x_i - y_j}(\partial_{x_i} - k\partial_{y_j}) + \frac{x_i y_j + 1}{x_i y_j - 1}(\partial_{x_i} + k\partial_{y_j}) \right),$$

where $\partial_{x_i} = x_i \frac{\partial}{\partial x_i}$, $\partial_{y_j} = y_i \frac{\partial}{\partial y_j}$ and the parameters k, p, q, r, s satisfy the relations $p = kr$, $2q + 1 = k(2s + 1)$. In the formulae below we always suppose that $h = -km - n - \frac{1}{2}p - q$ where m, n are non negative integer numbers. In order to define coefficients of the Pieri formulae let us introduce the flowing notations: let $H(m, n)$ be the set of partitions λ such that $\lambda_{m+1} \leqslant n$ and

$$n(\lambda) = \lambda_2 + 2\lambda_3 + \dots, \qquad c_\lambda = 2n(\lambda') + 2kn(\lambda) + |\lambda|(2h + 2k + 1),$$
$$a_i = \lambda_i + ki, \quad i = 1, 2, \dots, \qquad c_\lambda^0(\Box, x) = j - 1 + k(i - 1) + x,$$
$$c_\lambda^-(\Box, x) = \lambda_i - j - k(\lambda_j' - i) + x, \qquad c_\lambda^+(\Box, x) = \lambda_i + j + k(\lambda_j' + i) + x,$$

where λ' denote the conjugate partition to λ and $\Box$ denote the box (ij) (i is the row index and j is the column index) of the Young diagram which corresponding partition is λ (see [4]). Let us set

$$C_\lambda^0(x) = \prod_{\Box \in \lambda} c_\lambda^0(\Box, x), \quad C_\lambda^-(x) = \prod_{\Box \in \lambda} c_\lambda^-(\Box, x), \quad C_\lambda^+(x) = \prod_{\Box \in \lambda} c_\lambda^+(\Box, x),$$
$$J_\lambda(1) = 4^\lambda \frac{C_\lambda^0(h + \frac{1}{2}p + q)}{C_\lambda^-(-k)} \frac{C_\lambda^0(k + h - \frac{1}{2}p + \frac{1}{2})}{C_\lambda^+(2h - 1)}.$$

Let us also denote by $S^+(\lambda)$ the set of partitions μ which can be obtained from λ by adding one box and by $S^-(\lambda)$ we will denote the set of partitions μ which can be obtained from λ by deleting one box. Let us also set $S(\lambda) = S^+(\lambda) \cup S^-(\lambda) \cup \{\lambda\}$.

If $\mu \in S(\lambda)^+$ and $\mu_i = \lambda_i + 1$ then we set

$$
V_\mu(\lambda) = \prod_{\substack{j \neq i}}^{l(\lambda)+1} \frac{(a_i - a_j - k)}{(a_i - a_j)} \frac{(a_i + a_j + 2h - k)}{(a_i + a_j + 2h)} \times
$$

$$
\times \frac{(a_i - k + h + \frac{1}{2}p + q)}{(a_i - k(l(\lambda) + 2))} \frac{(a_i + k(l(\lambda) + 1) + 2h)}{(a_i + h)} \frac{(a_i + h - \frac{1}{2}p + \frac{1}{2})}{(a_i + h + \frac{1}{2})}.
$$

If $\mu \in S^-(\lambda)$ and $\mu_i = \lambda_i - 1$ then we set

$$
V_\mu(\lambda) = \prod_{\substack{j \neq i}}^{l(\lambda)} \frac{(a_i - a_j + k)}{(a_i - a_j)} \frac{(a_i + a_j + k + 2h)}{(a_i + a_j + 2h)} \times
$$

$$
\times \frac{(a_i + k + h - \frac{1}{2}p - q)}{(a_i + k(l(\lambda) + 1) + 2h)} \frac{(a_i - kl(\lambda))}{(a_i + h)} \frac{(a_i + h + \frac{1}{2}p - \frac{1}{2})}{(a_i + h - \frac{1}{2})},
$$

and

$$
a_{\lambda,\mu} = V_\mu(\lambda) \frac{J_\lambda(1)}{J_\mu(1)}, \quad \mu \in S(\lambda) \setminus \{\lambda\}, \quad a_{\lambda,\lambda} = -k^{-1}(2h + p + 2q) - \sum_{\mu \in S(\lambda)\setminus\{\lambda\}} V_\mu(\lambda).
$$

Let $P_{n,m} = \mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ be the algebra of Laurent polynomials in $m + n$ indeterminates. Now we are ready to define super Jacobi polynomials.

Theorem 1. *Let $ak + bh + c \neq 0$ for any $a, b, c \in \mathbb{Z}$. Then there exists a unique family of polynomials $J_\lambda \in P_{n,m}, \lambda \in H(m, n)$ such that:*

$$
J_\varnothing = 1, \quad \mathscr{L}J_\lambda = c_\lambda J_\lambda, \quad p_1 J_\lambda = \sum_{\mu \in S(\lambda)} a_{\lambda,\mu} J_\mu, \tag{1}
$$

where $p_1 = x_1 + x_1^{-1} + \cdots + x_m + x_m^{-1} + k^{-1}(y_1 + y_1^{-1} + \cdots + y_n + y_n^{-1})$.

Proof. Let $ak + bh + c \neq 0$ for any $a, b, c \in \mathbb{Z}$. Then it is not difficult to verify that from the conditions $c_\nu = c_\mu$ and $\mu, \nu \in S(\lambda)$ it follows that $\mu = \nu$. Therefore the operator

$$
\mathscr{L}_\mu^\lambda = \prod_{\nu \in S(\lambda)\setminus\{\mu\}} \frac{\mathscr{L} - c_\nu}{c_\mu - c_\nu},
$$

is well defined. So if a family of polynomials $\{J_\lambda\}$ satisfy the conditions of the Theorem then from the last formula in (1) it follows that

$$
\mathscr{L}_\mu^\lambda(p_1 J_\lambda) = a_{\lambda,\mu} J_\mu, \tag{2}
$$

and uniqueness can be proved by induction on the number of boxes in μ. Existence follows from [7, sec. 7]. $\qquad\square$

2. SPECIALISATION OF THE COEFFICIENTS OF PIERI FORMULAE

Limit as $k \to -1$

For a function $F(k)$ let us define $\tilde{F} = \lim_{k \to -1} F(k)$. For example $\lim_{k \to -1} a_{\lambda,\mu} = \tilde{a}_{\lambda,\mu}$.

Lemma 1. 1. *Let λ be a Young diagram such that we can add a box to the i-th row. Then the following equalities hold true*

$$\prod_{s=1}^{\lambda_i} \frac{x + \lambda'_s - s + 1}{x + \lambda'_s - s} = \prod_{t=i}^{l(\lambda)} \frac{x - \lambda_{t+1} + t}{x - \lambda_t + t}, \tag{3}$$

$$\prod_{s=1}^{\lambda_i} \frac{x - \lambda'_s + s}{x - \lambda'_s + s - 1} = \prod_{t=i}^{l(\lambda)} \frac{x + \lambda_t - t}{x + \lambda_{t+1} - t}. \tag{4}$$

2. *Let λ be a Young diagram such that we can delete a box from the i-th row. Then the following equalities hold true*

$$\prod_{s=1}^{\lambda_i - 1} \frac{x + \lambda'_s - s}{x + \lambda'_s - s + 1} = \frac{x - \lambda_i + i + 1}{x + l(\lambda)} \prod_{t=i+1}^{l(\lambda)} \frac{x - \lambda_t + t}{x - \lambda_t + t - 1}, \tag{5}$$

$$\prod_{s=1}^{\lambda_i - 1} \frac{x - \lambda'_s + s - 1}{x - \lambda'_s + s} = \frac{x - l(\lambda)}{x + \lambda_i - i - 1} \prod_{t=i+1}^{l(\lambda)} \frac{x + \lambda_t - t + 1}{x + \lambda_t - t}. \tag{6}$$

Proof. Let us prove (3). If $\lambda_i = 0$ then $i = l(\lambda) + 1$ and the both sides of equality (3) are equal to 1. For any integer a, b we will denote by $[[a, b]]$ the set on integers in the segment $[a, b]$. Suppose that $\lambda_i > 0$. Let us represent the segment $[[1, \lambda_i]]$ as the disjoint union

$$[[1, \lambda_i]] = \bigcup_{t=i}^{l(\lambda)} \Delta_t,$$

where $\Delta_t = [[\lambda_{t+1} + 1, \lambda_t]]$.

Therefore

$$\prod_{s=1}^{\lambda_i} \frac{x + \lambda'_s - s + 1}{x + \lambda'_s - s} = \prod_{t=i}^{l(\lambda)} \prod_{s \in \Delta_t} \frac{x + \lambda'_s - s + 1}{x + \lambda'_s - s}.$$

Let us check that

$$\prod_{s \in \Delta_t} \frac{x + \lambda'_s - s + 1}{x + \lambda'_s - s} = \frac{x - \lambda_{t+1} + t}{x - \lambda_t + t}. \tag{7}$$

Indeed if $\lambda_{t+1} = \lambda_t$ then $\Delta_t = \varnothing$ and the left hand side of the equality is 1 by definition. The right hand side is also 1.

If $\lambda_{t+1} < \lambda_t$ then for any $s \in \Delta_t$ we have $\lambda'_s = t$ and equality (7) is easy to verify. So we have proved the first equality. The second equality follows from the first one by the substitution $x \to -x$. Equalities (5), (6) can be proved in the same way. $\qquad \square$

Now we are going to prove our main result of this section.

Theorem 2. *The following statements hold true*
1) *if $\mu \in S^+(\lambda)$ then $\tilde{a}_{\lambda,\mu} = 1$;*

2) *if* $\mu \in S^-(\lambda)$ *and* $\mu_i = \lambda_i - 1$ *then*

$$\tilde{a}_{\lambda,\mu} = \frac{(2\tilde{a}_i + 2\tilde{h} + p + 2q)(2\tilde{a}_i + 2\tilde{h} - p - 2q - 2)}{(2\tilde{a}_i + 2\tilde{h} - 1)^2} \frac{(2\tilde{a}_i + 2\tilde{h} + p - 1)(2\tilde{a}_i + 2\tilde{h} - p - 1)}{(2\tilde{a}_i + 2\tilde{h})(2\tilde{a}_i + 2\tilde{h} - 2)};$$

3) $\tilde{a}_{\lambda,\lambda} = \dfrac{p(p + 2q + 1)}{2\tilde{h} - 2l(\lambda) - 1} + p - \displaystyle\sum_{i=1}^{l(\lambda)} \dfrac{2p(p + 2q + 1)}{(2\tilde{a}_i + 2\tilde{h} - 1)(2\tilde{a}_i + 2\tilde{h} + 1)}.$

Proof. Let us prove the statement 1). Let us represent $V_\mu(\lambda)$ as the product $V_\mu(\lambda) = V_\mu(\lambda, 1)V_\mu(\lambda, 2)$, where $V_\mu(\lambda, 1)$ is the part of $V_\mu(\lambda)$ which does not depend on h and $V_\mu(\lambda, 2)$ is the part which depends on h. The same notation $J_{\lambda,1}(1)$, $J_{\lambda,2}(1)$ will be used for $J_\lambda(1)$. We are going to prove two equalities

$$\lim_{k \to -1} V_\mu(\lambda, 1)\frac{J_{\lambda,1}(1)}{J_{\mu,1}(1)} = 1, \quad \lim_{k \to -1} V_\mu(\lambda, 2)\frac{J_{\lambda,2}(1)}{J_{\mu,2}(1)} = 1. \tag{8}$$

Let us prove the first one. It can be rewritten in the form:

$$\prod_{r \neq i}^{l(\lambda)+1} \frac{(\tilde{a}_i - \tilde{a}_r + 1)}{(\tilde{a}_i - \tilde{a}_r)} \frac{1}{(\tilde{a}_i + l(\lambda) + 2)} \frac{C_\mu^-(1)}{C_\lambda^-(1)} = 1,$$

where (ij) is the added box.

It is easy to check the following equality

$$\frac{C_\mu^-(1)}{C_\lambda^-(1)} = \prod_{s=1}^{j-1} \frac{\tilde{a}_i + \lambda'_s - s + 2}{\tilde{a}_i + \lambda'_s - s + 1}(\tilde{a}_i + \lambda'_j - j + 3)\prod_{r=1}^{i-1} \frac{\tilde{a}_r - j + \lambda'_j + 2}{\tilde{a}_r - j + \lambda'_j + 1}.$$

Since the box (ij) can be added to λ then we have $\lambda_i = j - 1$ and $\lambda'_j = i - 1$. Therefore

$$\prod_{r=1}^{i-1} \frac{\tilde{a}_r - j + \lambda'_j + 2}{\tilde{a}_r - j + \lambda'_j + 1} = \prod_{r=1}^{i-1} \frac{\tilde{a}_i - \tilde{a}_r}{\tilde{a}_i - \tilde{a}_r + 1}.$$

Besides we have $(\tilde{a}_i + \lambda'_j - j + 3) = 1$ and if we substitute $x = \tilde{a}_i + 1$ in the equality (3) then we get

$$\prod_{s=1}^{j-1} \frac{\tilde{a}_i + \lambda'_s - s + 2}{\tilde{a}_i + \lambda'_s - s + 1} = \prod_{t=i}^{l(\lambda)} \frac{\tilde{a}_i - \tilde{a}_{t+1}}{\tilde{a}_i - \tilde{a}_t + 1} = (\tilde{a}_i + l(\lambda) + 2)\prod_{t=i+1}^{l(\lambda)+1} \frac{\tilde{a}_i - \tilde{a}_t}{\tilde{a}_i - \tilde{a}_t + 1}.$$

Therefore we have proved the first equality in (8). By using the identity (4) we can prove in the same way the second equality in (8) and therefore we have proved the statement 1) of the Theorem.

The statement 2) can be proved analogously using the identities (5) and (6).

Now let us prove the statement 3). Consider the expansion into partial fractions of the function $V_\mu(h)$. If $\mu \in S(\mu)^+$ and $\mu_i = \lambda_i + 1$ then

$$V_\mu(\lambda) = Q_i + \frac{B_i^+}{\tilde{a}_i + h} + \frac{C_i^+}{\tilde{a}_i + \frac{1}{2} + h} + \sum_{j \neq i}^{l(\lambda)+1} \frac{A_{ij}^+}{\tilde{a}_i + \tilde{a}_j + 2h},$$

where Q_i is a polynomial. In the same way if $\mu \in S(\lambda)^-$ and $\mu_i = \lambda_i - 1$ then

$$V_\mu(\lambda) = \frac{B_i^-}{\tilde{a}_i + h} + \frac{C_i^-}{\tilde{a}_i - \frac{1}{2} + h} + \frac{D_i^-}{\tilde{a}_i - l(\lambda) - 1 + 2h} + \sum_{j \neq i}^{l(\lambda)} \frac{A_{ij}^-}{\tilde{a}_i + \tilde{a}_j + 2h},$$

It is not difficult to verify the following equalities

$$A_{ij}^+ + A_{ji}^- = 0, \quad 1 \leqslant i \neq j \leqslant l(\lambda), \quad B_{l(\lambda)+1}^+ = 0, \quad C_{l(\lambda)+1}^+ = -\frac{1}{2}p(p + 2q + 1),$$

$$D_i^- + A_{l(\lambda)+1,i} = 0, \quad B_i^+ + B_i^- = 0, \quad C_i^- = -C_i^+ = \frac{1}{2}p(p + 2q + 1), \quad 1 \leqslant i \leqslant l(\lambda).$$

Besides we have $Q_i = V_\mu(\lambda, 1)(2h + 2q + a_i + 1)$ and it can be checked that

$$\sum_{i=1}^{l(\lambda)+1} \tilde{a}_i \tilde{V}_i^+(\lambda, 1) = -1, \quad \sum_{i=1}^{l(\lambda)+1} \tilde{V}_i^+(\lambda, 1) = -1.$$

This proves statement 3) and the Theorem. □

Limit as $(p, q) \to (-1, 0)$

Let us denote $\lim_{(p,q) \to (-1,0)} \tilde{F}$ by $F^\sharp$.

Theorem 3. *The following statements hold true*
1) *if* $\mu \in S^+(\lambda)$ *then* $a_{\lambda,\mu}^\sharp = 1$;
2) *if* $\mu \in S^-(\lambda)$ *and* $\mu_i = \lambda_i - 1$ *then* $a_{\lambda,\mu}^\sharp = 1 - \delta(a_i + d)$;
3) $a_{\lambda,\lambda}^\sharp = \sum_{i=1}^{l(\lambda)}(\delta(a_i + d + 1) - \delta(a_i + d)) - 1 + \delta(d - l(\lambda))$, *where* $d = m - n$ *and* $\delta(x) = 0$
if $x \neq 0$ *and* $\delta(x) = 1$ *if* $x = 0$.

A proof of the Theorem easily follows from the Theorem 2.

3. SPECIALISATION OF THE SUPER JACOBI POLYNOMIALS

In the previous section we specialised the coefficients of the Pieri formula. Now we can prove an existence of specialised super Jacobi polynomials and the corresponding Pieri formulae.

Specialisation as $k \to -1$

Theorem 4. *Let* $p + 2q \notin \mathbb{Q}$ *then there exists* $\lim_{k \to -1} J_\lambda = \tilde{J}_\lambda$ *and the following Pieri formulae hold true*

$$\tilde{p}_1 \tilde{J}_\lambda = \sum_{\mu \in S(\lambda)} \tilde{a}_{\lambda,\mu} \tilde{J}_\mu. \tag{9}$$

Proof. Suppose that $p + 2q \notin \mathbb{Q}$ then it is not difficult to verify that if $\tilde{c}_\nu = \tilde{c}_\mu$ and $\mu, \nu \in S(\lambda)$ then $\mu = \nu$. Therefore there exists a limit

$$\lim_{k \to -1} \mathscr{L}_\mu^\lambda = \tilde{\mathscr{L}}_\mu^\lambda = \prod_{\nu \in S(\lambda) \backslash \{\mu\}} \frac{\tilde{\mathscr{L}} - \tilde{c}_\nu}{\tilde{c}_\mu - \tilde{c}_\nu}.$$

Now let prove the existence of $\lim_{k \to -1} J_\lambda$ by induction on $|\lambda|$. If $|\lambda| = 0$, then $J_\varnothing = 1$ and the existence is clear. Let $|\lambda| > 0$ and μ can be obtained from λ by deleting one box. Then from formula (2) and statement 1) of Theorem 2 it follows that $\lim_{k \to -1} J_\lambda = \tilde{J}_\lambda$ does exist and the following equality holds true

$$\tilde{\mathscr{L}}_\mu^\lambda(\tilde{p}_1 \tilde{J}_\mu) = \tilde{a}_{\mu,\lambda} \tilde{J}_\lambda. \tag{10}$$

So the equality (9) can be obtained from (1) by taking the limit as $k \to -1$. $\qquad\square$

Specialisation as $(p, q) \to (-1, 0)$

Let us prove the existence $\lim_{(p,q) \to (-1,0)} \tilde{J}_\lambda = J_\lambda^\sharp$. In this case the equation $c_\mu^\sharp = c_\nu^\sharp$ can have more than one solution for $\mu, \nu \in S(\lambda)$. The following Lemma is not difficult to prove.

Lemma 2. *Let $\mu, \nu \in S(\lambda)$ and $\mu \neq \nu$ then $c_\mu^\sharp = c_\nu^\sharp$ if and only if one of the following conditions are fulfilled:*
1) up to a permutation $\mu = \lambda$, $\nu = \lambda \cup \square$ and $c_\lambda^0(\square, d) = 0$;
2) up to a permutation $\mu = \lambda$, $\nu = \lambda \setminus \square$ and $c_\lambda^0(\square, d) = 0$;
3) there exists $i \in \mathbb{Z}$, $i \neq 0$ such that up to a permutation $\mu = \lambda \cup \square$ and $\nu = \lambda \setminus \tilde{\square}$ где $c^0(\square, d) = i$, $c^0(\tilde{\square}, d) = -i$.

So the previous Lemma shows that the equation $c_\mu^\sharp = c_\nu^\sharp$ can have not more than two solutions for $\mu, \nu \in S(\lambda)$. For this case we need the following Lemma.

Lemma 3. *Let $c_\mu^\sharp = c_\nu^\sharp = \gamma$ for some $\mu \neq \nu \in S(\lambda)$. Then*

$$\lim_{(p,q) \to (-1,0)} (\tilde{\mathscr{L}}_\nu^\lambda + \tilde{\mathscr{L}}_\mu^\lambda) = \left[(\gamma - \mathscr{L}^\sharp) \frac{f'(\gamma)}{f(\gamma)^2} + \frac{1}{f(\gamma)} \right] f(\mathscr{L}^\sharp), \text{ where } f(t) = \prod_{\tau \in S(\lambda) \setminus \{\mu, \nu\}} (t - c_\tau).$$

Let us denote the limit above by $\mathscr{L}_{\mu,\nu}^\sharp$. Now we can prove the main result of this section.

Theorem 5. *There exists a unique family of polynomials $J_\lambda^\sharp \in P_{m,n}$, $\lambda \in H(m,n)$ such that*

$$J_\varnothing^\sharp = 1, \quad \mathscr{L}^\sharp J_\lambda^\sharp = c_\lambda^\sharp J_\lambda^\sharp, \quad p_1^\sharp J_\lambda^\sharp = \sum_{\mu \in S(\lambda)} a_{\lambda,\mu}^\sharp J_\mu^\sharp, \tag{11}$$

where $p_1^\sharp = x_1 + x_1^{-1} + \cdots + x_m + x_m^{-1} - (y_1 + y_1^{-1} + \cdots + y_n + y_n^{-1})$.

Proof. Let us prove an existence first. Let μ be a non empty Young diagram and λ can be obtained from μ by deleting one box. Suppose that $c_\nu^\sharp \neq c_\mu^\sharp$ for all $\nu \in S(\lambda) \setminus \{\mu\}$. Let us take the limit as $(p, q) \to (-1, 0)$ in the equality (10). Then we get

$$\prod_{\nu \in S(\lambda) \setminus \{\mu\}} \frac{\mathscr{L}^\sharp - c_\nu^\sharp}{c_\mu^\sharp - c_\nu^\sharp} (p_1^\sharp J_\lambda^\sharp) = J_\mu^\sharp. \tag{12}$$

This proves the existence of J_μ in this case. Suppose now that the equation $c_\nu^\sharp = c_\mu^\sharp$ has two different solutions $\mu, \nu \in S(\lambda)$. Then by Lemma 3 we have

$$\mathscr{L}_{\mu,\nu}^\sharp(p_1^\sharp J_\lambda^\sharp) = J_\mu^\sharp + a_{\lambda,\nu}^\sharp J_\nu^\sharp. \tag{13}$$

By Lemma 2 we have $|\nu| < |\mu|$ therefore $J_\mu^\sharp$ is well defined and we proved the existence. Uniqueness follows from the formulae (12), (13). $\qquad\square$

4. EULER SUPERCHARACTERS

In this section we prove two main results using an explicit formula for Euler super-character by V. Serganova [9, 10]. The first one is Pieri formulae and the second result is that Euler supercharacters are eigenfunctions of the deformed CMS operator. As a corollary for any partition $\lambda \in H(m, n)$ we get the coincidence (up to sign) of the Euler supercharacter E_λ and the specialised super Jacobi polynomial $J_\lambda^\sharp$.

In order to define Euler supercharacters let us introduce new variables $u = x + x^{-1}$, $v = y + y^{-1}$ and the following polynomials in one variable z

$$\varphi_a(z) = \frac{w^{a+\frac{1}{2}} - w^{-a-\frac{1}{2}}}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}}, \quad \psi_a(z) = \frac{w^{a+\frac{1}{2}} + w^{-a-\frac{1}{2}}}{w^{\frac{1}{2}} + w^{-\frac{1}{2}}}, \quad z = w + w^{-1},$$

where a is integer.

It it easy to check that these polynomials satisfiy the following recurrence relations

$$z\varphi_i = \varphi_{i+1} + \varphi_{i-1}, \ \varphi_0 = 1, \ \varphi_{-i} = -\varphi_{i-1}, \tag{14}$$
$$z\psi_i = \psi_{i+1} + \psi_{i-1}, \ \psi_0 = 1, \ \psi_{-i} = \psi_{i-1}. \tag{15}$$

Let us also for $\lambda \in H(m, n)$ define $d = m - n$ and the following numbers

$$r_i = \begin{cases} \lambda_i + d - i, & 1 \leqslant i \leqslant i(\lambda), \\ m - i, & i(\lambda) < i \leqslant m, \end{cases} \quad s_j = \begin{cases} \lambda'_j - d - j, & 1 \leqslant j \leqslant j(\lambda), \\ n - j, & j(\lambda) < j \leqslant n, \end{cases}$$

where

$$i(\lambda) = \max\{0, \, i \mid \lambda_i + d - i \geqslant 0, \quad 1 \leqslant i \leqslant m\}, \quad j(\lambda) = \max\{0, \, j \mid \lambda'_j - d - j \geqslant 0, \quad 1 \leqslant j \leqslant n\}.$$

Let us also denote by π_λ the set of pairs (i, j) such that $i \leqslant i(\lambda)$ or $j \leqslant j(\lambda)$ and set

$$\Pi_\lambda(u, v) = \prod_{(ij) \in \pi_\lambda} (u_i - v_j), \quad \Delta(u) = \prod_{i<j}(u_i - u_j), \quad \Delta(v) = \prod_{i<j}(v_i - v_j),$$
$$\varphi_\lambda(u) = \varphi_{r_1}(u_1) \ldots \varphi_{r_m}(u_m), \quad \psi_\lambda(v) = \psi_{s_1}(v_1) \ldots \psi_{s_n}(v_n),$$

and for function $f(u, v)$ define the alternation operation

$$\{f(u, v)\} = \sum_{\sigma \in S_m \times S_n} \varepsilon(\sigma)\sigma(f(u, v)).$$

The following formula for Euler supercharacters in variables u, v can be easily deduced from the last formula for Euler supecharacter in [8, p. 4307]

$$\Delta(u)\Delta(v)E_\lambda = (-1)^{t(\lambda)}\{\Pi_\lambda(u, v)\varphi_\lambda(u)\psi_\lambda(v)\}, \tag{16}$$

where $t(\lambda)$ equals to the number of pairs $(i, j) \in \pi_\lambda$ such that $j - i + d < 0$.

Theorem 6. 1. *Euler supercharacters E_λ satisfy the following Pieri formula*

$$p_1^\sharp E_\lambda = \sum_{\mu \in S(\lambda)} (-1)^{s(\lambda)-s(\mu)} a_{\lambda,\mu}^\sharp E_\mu,$$

where $s(\lambda)$ equals to the number of pairs $(ij) \in \lambda \cup \pi_\lambda$ such that $i - j > d$.

2. *Euler supecharacter E_λ is an eigenfunction of the deformed CMS operator $\mathscr{L}^\sharp$ with eigenvalue $c_\lambda^\sharp$.*

Proof. By $\lambda + \varepsilon_i$ (and the same for $\lambda - \varepsilon_i$) we will denote the diagram obtaining from λ by adding one box to the i-th row (or deleting one box from the i-th row). The same notations $\lambda \pm \delta_j$ will be used for columns. Let us also denote by F_λ the product $\Pi_\lambda(u, v)\varphi_\lambda(u)\psi_\lambda(v)$. We are going to calculate $\{u_i F_\lambda\}$ and $\{v_j F_\lambda\}$. The following equalities hold true

$$\{u_i F_\lambda\} = \begin{cases} \{F_{\lambda+\varepsilon_i}\} + \{F_{\lambda-\varepsilon_i}\}, & i < i(\lambda), \\ 0, & i(\lambda) + 1 < i < m, \\ -\{F_\lambda\}, & i(\lambda) + 1 < i = m, \end{cases}$$

$$\{v_j F_\lambda\} = \begin{cases} \{F_{\lambda+\delta_j}\} + \{F_{\lambda-\delta_j}\}, & j < j(\lambda), \\ 0, & j(\lambda) + 1 < j < n, \\ \{F_\lambda\}, & j(\lambda) + 1 < j = n, \end{cases}$$

if $i = i(\lambda)$, $j = j(\lambda)$ then

$$\{u_i F_\lambda\} = \begin{cases} \{F_{\lambda+\varepsilon_i}\} + \{F_{\lambda-\varepsilon_i}\}, & r_i > 0, \\ \{F_{\lambda+\varepsilon_i}\} - \{F_\lambda\}, & r_i = 0, \end{cases}$$

$$\{v_j F_\lambda\} = \begin{cases} \{F_{\lambda+\delta_j}\} + \{F_{\lambda-\delta_j}\}, & s_j > 0, \\ \{F_{\lambda+\delta_j}\} + \{F_\lambda\}, & s_j = 0, \end{cases}$$

if $i = i(\lambda) + 1$, $j = j(\lambda) + 1$ then

$$\{(u_i - v_j)F_\lambda\} = \begin{cases} 2\{F_\lambda\} \text{ if } i < m \text{ and } r_{i-1} = 0 \text{ or } s_{j-1} = 0, \\ 0 \text{ if } i = m \text{ and } r_{i-1} = 0 \text{ or } s_{j-1} = 0, \\ 2\{F_\lambda\} + (-1)^{m-i+1}\{F_{\lambda+\varepsilon_i}\} \text{ if } i < m \text{ and } r_{i-1} > 0, \ s_{j-1} > 0, \\ -\{F_{\lambda+\varepsilon_i}\} \text{ if } i = m \text{ and } r_{i-1} > 0, \ s_{j-1} > 0. \end{cases} \quad (17)$$

All the above equalities except the equalities (17) can be easily proved from the definition of the Euler supercharacters. So let us prove the first equality in (17). It is easy to check that $m - i(\lambda) = n - j(\lambda)$. Therefore it is enough to prove this equality in the case when λ is a row or a column and $n = m$. Let $\lambda = (a)$, $a > 0$ be a row. Therefore $\Pi_\lambda(u, v) = (u_1 - v_1)\prod_{j=2}^n(u_1 - v_j)\prod_{i=2}^n(u_i - v_2)$ and we need to prove that

$$\{(u_2 - v_2 - 2)\Pi_\lambda(u, v)\varphi_{a-1}(u_1)\varphi_{n-2}(u_2)\ldots\varphi_0(u_n)\psi_{n-2}(v_2)\ldots\psi_0(v_n)\} = 0.$$

Using the recurrence relations (14), (15) we can rewrite the previous equality in the form

$$\{(u_2 - v_2)\Pi_\lambda(u, v)\varphi_{a-1}(u_1)u_2^{n-2}u_3^{n-2}\ldots u_n^0 v_2^{n-2}v_3^{n-3}\ldots v_n^0\} = 0.$$

Now using the alternation operation over subgroup $\{1\} \times S_n$ we can rewrite the previous equality in the form

$$\left\{\varphi_{a-1}(u_1)\left(\sum_{i=3}^n u_i\right)\prod_{j=1}^n(u_1 - v_j)u_2^{n-2}\ldots u_n^0 v_1^{n-1}v_2^{n-2}\ldots v_n^0\right\} = 0.$$

And in this form this equality can be easily checked. The other equalities in (17) can be proved in the same way.

Now let us prove the second statement of the Theorem. Denote by D the following expression

$$\prod_{i,j}(u_i - v_j)\left(\Delta(u)\Delta(v)\prod_{i=1}^{m}(x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}})\prod_{j=1}^{n}(y_i^{\frac{1}{2}} + y_j^{-\frac{1}{2}})\right)^{-1}$$

and rewrite the formula (16) in the following form

$$D^{-1}E_\lambda = (-1)^{t(\lambda)}\{F_\lambda\} = (-1)^{t(\lambda)}\left\{\frac{x_1^{\rho_1}\ldots x_m^{\rho_m}y_1^{\tau_1}\ldots y_n^{\tau_n}}{\prod_{(ij)\notin\pi_\lambda}(u_i - v_j)}\right\}. \tag{18}$$

It is known (see [11]) that $D\mathscr{L}^\sharp D^{-1} = \sum_{i=1}^{m}\partial_{x_i}^2 - \sum_{j=1}^{n}\partial_{y_j}^2 - (\rho_{m,n},\rho_{m,n})$, where by definition $(\rho_{m,n},\rho_{m,n}) = -\sum_{i=1}^{d}\left(d - i + \frac{1}{2}\right)^2$.

Let us note that if $n = m$, then $(\rho_{m,n},\rho_{m,n}) = 0$. So we see that it is enough to prove that F_λ is an eigenfunction of the operator $D\mathscr{L}^\sharp D^{-1}$ with eigenvalue $c_\lambda^\sharp$.

If $\lambda = \varnothing$ then it easily follows from the formula (16) that $E_\varnothing = 1$. Therefore $\mathscr{L}^\sharp(E_\varnothing) = 0$ and we arrived to an identity

$$\left(\sum_{i=1}^{m}\partial_{x_i}^2 - \sum_{j=1}^{n}\partial_{y_j}^2\right)\{F_\varnothing\} = (\rho_{m,n},\rho_{m,n})\{F_\varnothing\}.$$

Using this identity it is not difficult to see that

$$\left(\sum_{i=1}^{m}\partial_{x_i}^2 - \sum_{j=1}^{n}\partial_{y_j}^2\right)\{F_\lambda\} = \left(\sum_{i=1}^{i(\lambda)}r_i^2 - \sum_{j=1}^{j(\lambda)}s_j^2\right)\{F_\lambda\}. \tag{19}$$

So we see from the formula (19) that E_λ is an eigenfunction of $\mathscr{L}^\sharp$ with eigenvalue

$$\sum_{i=1}^{i(\lambda)}\left(\lambda_i + d - i + \frac{1}{2}\right)^2 - \sum_{j=1}^{j(\lambda)}\left(\lambda_i' - d - j + \frac{1}{2}\right)^2 - \sum_{i=1}^{d}\left(d - i + \frac{1}{2}\right)^2$$

and it is not difficult to verify that the last expression is $c_\lambda^\sharp$. Theorem is proved. $\square$

Corollary 1. *The following equality holds true*

$$J_\lambda^\sharp(u_1,\ldots,u_m,v_1,\ldots,v_n) = (-1)^{s(\lambda)}E_\lambda(u_1,\ldots,u_m,v_1,\ldots,\tilde{v}_n).$$

Proof. It is enough to apply Theorem 5 to the family E_λ. All the conditions of this Theorem are fulfilled except the condition $E_\lambda \in P_{m,n}$.

But this condition easily follows from the Pieri formula and induction on $|\lambda|$. $\square$

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Формула Пиери и специализация супермногочленов Якоби

А. Н. Сергеев, Е. Д. Жаринов

Сергеев Александр Николаевич, доктор физико-математических наук, профессор кафедры геометрии, Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, г. Саратов, ул. Астраханская, д. 83, SergeevAN@info.sgu.ru, asergeev@hse.ru

Жаринов Егор Дмитриевич, аспирант кафедры геометрии, Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, г. Саратов, ул. Астраханская, д. 83, egor.zharinov@gmail.com

Ранее было доказано, что суперхарактеры Эйлера супералгебры Ли $\mathfrak{osp}(2m+1, 2n)$ являются предельным случаем супермногочленов Якоби. Этот результат был первым примером, показывающим, какого рода связи возникают между собственными функциями деформированных операторов Калоджеро – Мозера – Сазерленда и теорией представлений. К сожалению, доказательство этого результата было чисто вычислительным. В данной работе мы предлагаем более простое и концептуальное доказательство, основная идея которого заключается в использовании с самого начала формулы Пиери. Мы надеемся, что наш подход окажется полезным во многих аналогичных ситуациях.

Ключевые слова: квантовый оператор CMS, формула Пьери, супермногочлены Якоби, супералгебра, суперхарактер Эйлера.

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