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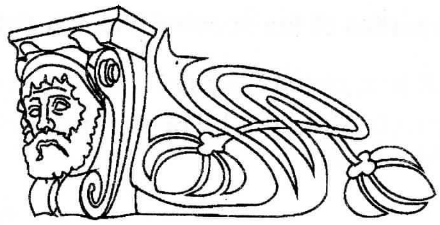

ВОССТАНОВЛЕНИЕ СИНГУЛЯРНЫХ ДИФФЕРЕНЦИАЛЬНЫХ ПУЧКОВ С ТОЧКАМИ ПОВОРОТА

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Рассматриваются пучки дифференциальных уравнений 2-го порядка на полуоси с точками поворота. Устанавливаются свойства спектра и исследуется обратная спектральная задача восстановления коэффициентов пучка по спектральным данным.

Recovering singular differential pencils with a turning point

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Second-order pencils of differential equations on the half-line with turning points are considered. We establish properties of the spectrum and study the inverse spectral problem of recovering coefficients of the pencil from the spectral data.

1. Introduction

The paper deals with the singular indefinite non-selfadjoint boundary value problem L for the differential equation

$$y''(x)+\left(\rho^2 r(x)+i\rho q_1(x)+q_0(x)\right)y(x)=0, \quad x\ge 0, \tag{1}$$

on the half-line with the boundary condition

$$U(y):=y'(0)+(\beta_1\rho+\beta_0)y(0)=0, \tag{2}$$

where ρ is the spectral parameter. Let a, $\omega>0$, and let $r(x)=-\omega^2$ for $x\in[0,a]$ and $r(x)=1$ for $x>a$, i.e. the weight-function $r(x)$ changes the sign in an interior point, which is called the turning point. The functions $q_j(x)$ are complex-valued, $q_1(x)$ is absolutely continuous, and $(1+x)q_j^{(\nu)}(x)\in L(0,\infty)$ for $0\le\nu\le j\le 1$. The coefficients β_1 and β_0 are complex numbers and $\beta_1\ne\pm\omega$. The last condition excludes from the consideration Regge-type problems which require a separate investigation.

Differential equations with nonlinear dependence on the spectral parameter and with turning points arise in various problems of mathematics as well as in applications (see [1]-[9] for details). In this paper we establish properties of the spectrum of the boundary value problem L and study the inverse problem for singular non-selfadjoint indefinite pencil (1). Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. For the classical Sturm-Liouville operator the inverse problem has been studied fairly completely (see [10]-[12] and the references therein). Some aspects of the inverse problem theory for differential pencils without turning points were studied in [13]-[19] and other works. In [20]-[25] the inverse problem was investigated for differential equations with turning points but with linear dependence on the spectral parameter.

Indefinite differential pencils with turning point produce essential qualitative modifications in the investigation of the inverse problem. To study the inverse problem of this kind we use the method of spectral mappings [26] connected with ideas of the contour integral method. In Section 2 we obtain properties of the spectral characteristics of the boundary value problem L. In Section 3 we give a formulation of the inverse problem and prove the uniqueness theorem for the solution of this inverse problem. A constructive procedure for the solution of the inverse problem will be given in a separate paper.

2. Properties of the spectral characteristics

Let $\varphi(x,\rho)$ and $S(x,\rho)$ be solutions of equation (1) under the initial conditions $\varphi(0,\rho)=1$, $U(\varphi)=0$, $S(0,\rho)=0$, $S'(0,\rho)=1$. For each fixed $x\ge 0$, the functions $\varphi^{(m)}(x,\rho)$ and $S^{(m)}(x,\rho)$, $m=0,1$ are entire in ρ. Moreover,

$$\langle \varphi(x,\rho), S(x,\rho)\rangle \equiv 1, \tag{3}$$

where $\langle y,z\rangle := yz' - y'z$, since by virtue of Liouville's formula for the Wronskian [27] $\langle \varphi(x,\rho), S(x,\rho)\rangle$ does not depend on x.

Denote $\Pi_\pm := \{\rho:\ \pm\operatorname{Im}\rho>0\}$, $\Pi_0 := \{\rho:\ \operatorname{Im}\rho=0\}$. By the well-known method (see, for example, [1], [28]-[29]) we get that for $x\ge a$, $\rho\in\overline{\Pi_+}$, there exists a solution $e(x,\rho)$ of equation (1) (which is called the Jost-type solution) with the following properties:

1^0. For each fixed $x\ge a$, the functions $e^{(m)}(x,\rho)$, $m=0,1$ are holomorphic for $\rho\in\Pi_+$ and $\rho\in\Pi_-$ (i.e. they are piecewise holomorphic).

2^0. The functions $e^{(m)}(x,\rho)$, $m=0,1$ are continuous for $x\ge a$, $\rho\in\overline{\Pi_+}$ and $\rho\in\overline{\Pi_-}$ (we differ the sides of the cut Π_0). In other words, for real ρ there exist the finite limits

$$e_\pm^{(m)}(x,\rho)=\lim_{z\to\rho,\,z\in\Pi_\pm} e^{(m)}(x,z).$$

Moreover the functions $e^{(m)}(x,\rho)$, $m=0,1$ are continuously differentiable with respect to $\rho\in\overline{\Pi_+}\setminus\{0\}$ and $\rho\in\overline{\Pi_-}\setminus\{0\}$.

3^0. For $x\to\infty$, $\rho\in\overline{\Pi_\pm}\setminus\{0\}$, $m=0,1$,

$$e^{(m)}(x,\rho)=(\pm i\rho)^m\exp(\pm(i\rho x-Q(x)))(1+o(1)), \tag{4}$$

where

$$Q(x)=\frac{1}{2}\int_0^x q_1(t)dt. \tag{5}$$

4^0. For $|\rho|\to\infty$, $\rho\in\overline{\Pi_\pm}$, $m=0,1$, uniformly in $x\ge a$.

$$e^{(m)}(x,\rho)=(\pm i\rho)^m\exp(\pm(i\rho x-Q(x)))[1], \tag{6}$$

where $[1]:=1+O(\rho^{-1})$.

We extend $e(x,\rho)$ to the segment $[0,a]$ as a solution of equation (1) which is smooth for $x\ge 0$ i.e.

$$e^{(m)}(a-0,\rho)=e^{(m)}(a+0,\rho),\quad m=0,1.. \tag{7}$$

Then the properties 1^0-2^0 remain true for $x\ge 0$.

The Jost-type solution $e(x,\rho)$, $x\ge 0$ is a generalization of the classical Jost solution for the Sturm-Liouville equation (see [10]-[12]).

Denote

$$\Delta(\rho):=U(e(x,\rho)). \tag{8}$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ there exist the finite limits

$$\Delta_\pm(\rho)=\lim_{z\to\rho,\,z\in\Pi_\pm}\Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho\in\overline{\Pi_{\pm}}\setminus\{0\}$..

Lemma 1. *For* $|\rho|\to\infty$, $\rho\in\overline{\Pi_{\pm}}$, *the following asymptotical formula holds*

$$\Delta(\rho)=\frac{\rho}{2}\exp\big(\pm(i\rho a-Q(a))\big)\big((\beta_1-\omega)(1\mp i/\omega)\exp(\omega\rho a-iQ(a)/\omega)[1]+$$
$$+(\beta_1+\omega)(1\pm i/\omega)\exp(-\omega\rho a+iQ(a)/\omega)[1]\big). \tag{9}$$

Proof. Denote $\Pi^1_{\pm}:=\{\rho:\pm\operatorname{Re}\rho>0\}$. Let $\{y_k(x,\rho)\}_{k=1,2}$, $x\in[0,a]$, $\rho_{\pm}\in\overline{\Pi^1}$, be the Birkhoff-type fundamental system of solutions of equation (1) on the interval $[0,a]$ with the asymptotics for $|\rho|\to\infty$, $m=0,1$:

$$y_k^{(m)}(x,\rho)=((-1)^k\omega\rho)^m\exp((-1)^k(\omega\rho x-iQ(x)/\omega))[1] \tag{10}$$

(see [1], [28]-[29]). Then

$$e(x,\rho)=b_1(\rho)y_1(x,\rho)+b_2(\rho)y_2(x,\rho),\quad x\in[0,a]. \tag{11}$$

Using (6), (7) and (10) we obtain for $\rho\in\overline{\Pi_{\pm}}$, $k=0,1$:

$$(-1)^k\exp(-\omega\rho a+iQ(a)/\omega)[1]b_1(\rho)+\exp(\omega\rho a-iQ(a)/\omega)[1]b_2(\rho)=$$
$$=(\pm i/\omega)^k\exp(\pm(i\rho a-Q(a)))[1].$$

Calculating $b_1(\rho)$ and $b_2(\rho)$ from this algebraic system and substituting the result into (11), we arrive at the following asymptotical formula for $|\rho|\to\infty$, $\rho\in\overline{\Pi_{\pm}}$, $m=0,1$, $x\in[0,a]$:

$$e^{(m)}(x,\rho)=\frac{(\omega\rho)^m}{2}\exp(\pm(i\rho a-Q(a)))\times$$
$$\times\big((-1)^m(1\mp i/\omega)\exp(\omega\rho a-iQ(a)/\omega)\exp(-\omega\rho x+iQ(x)/\omega)[1]+$$
$$+(1\pm i/\omega)\exp(-\omega\rho a+iQ(a)/\omega)\exp(\omega\rho x-iQ(x)/\omega)[1]\big). \tag{12}$$

Together with (2) and (8) this yields (9). Lemma 1 is proved. □

Similarly one can calculate

$$e(0,\rho)=\frac{1}{2}\exp(\pm(i\rho a-Q(a)))((1\mp i/\omega)\exp(\omega\rho a-iQ(a)/\omega)[1]+$$
$$+(1\pm i/\omega)\exp(-\omega\rho a+iQ(a)/\omega)[1]\big), \tag{13}$$

$$\dot{\Delta}(\rho)=\frac{\rho}{2}(\pm i\omega a^2)\exp\big(\pm(i\rho a-Q(a))\big)\big((\beta_1-\omega)(1\mp i/\omega)\exp(\omega\rho a-iQ(a)/\omega)[1]-$$
$$-(\beta_1+\omega)(1\pm i/\omega)\exp(-\omega\rho a+iQ(a)/\omega)[1]\big) \tag{14}$$

as $|\rho|\to\infty$, $\rho\in\overline{\Pi_{\pm}}$, where $\dot{\Delta}(\rho):=\dfrac{d}{d\rho}\Delta(\rho)$.

It follows from (9) that for sufficiently large $|\rho|$, the function $\Delta|\rho|$ has simple zeros of the form

$$\rho_k=\frac{1}{\omega a}(\kappa\pi i+iQ/\omega+\kappa\pm\kappa_1)+O\left(\frac{1}{\kappa}\right), \tag{15}$$

where $Q:=Q(a)$ and

$$\kappa = \frac{1}{2}\ln\frac{\beta_1 + \omega}{\beta_1 - \omega}, \quad \kappa_1 = \frac{1}{2}\ln\frac{i+\omega}{i-\omega}. \tag{16}$$

Here $\ln z := \ln|z| + i\arg z$, $\arg z \in [0, 2\pi)$. Denote

$$\Lambda'_{\pm} = \{\rho \in \Pi_{\pm} : \Delta(\rho) = 0\}, \quad \Lambda' = \Lambda'_{+} \cup \Lambda'_{-},$$

$$\Lambda''_{\pm} = \{\rho \in \mathbf{R} : \Delta_{\pm}(\rho) = 0\}, \quad \Lambda'' = \Lambda''_{+} \cup \Lambda''_{-},$$

$$\Lambda_{+} = \Lambda'_{+} \cup \Lambda''_{+}, \quad \Lambda = \Lambda_{+} \cup \Lambda_{-}.$$

Obviously $\Lambda = \Lambda' \cup \Lambda''$, Λ' is a countable unbounded set, and Λ'' is a bounded set.
We put

$$\Phi(x,\rho) = \frac{e(x,\rho)}{\Delta(\rho)}. \tag{17}$$

The function $\Phi(x,\rho)$ is a solution of equation (1), and on account of (2), (4) and (8), also the conditions $U(\Phi) = 1$, $\Phi(x,\rho) = O(\exp(\pm i\rho x))$, $x \to 0$, $\rho \in \Pi_{\pm}$ (while $\Delta(\rho) \neq 0$). In particular, $\lim_{x\to\infty} \Phi(x,\rho) = 0$. Denote

$$M(\rho) = \Phi(0,\rho). \tag{18}$$

We will call $M(\rho)$ the Weyl-type function for L since it is a generalization of the concept of the Weyl function for the classical Sturm-Liouville operator (see [30]). It follows from (17) and (18) that

$$M(\rho) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{19}$$

Using the conditions at the point $x = 0$ we get

$$\Phi(x,\rho) = S(x,\rho) + M(\rho)\varphi(x,\rho). \tag{20}$$

It follows from (3), (17) and (20) that

$$\langle \varphi(x,\rho), \Phi(x,\rho) \rangle \equiv 1, \tag{21}$$

$$\langle \varphi(x,\rho), e(x,\rho) \rangle \equiv \Delta(\rho). \tag{22}$$

Theorem 1. *The Weyl-type function $M(\rho)$ is holomorphic in $\Pi_{+} \setminus \Lambda'_{+}$ and continuously differentiable in $\overline{\Pi_{\pm}} \setminus \Lambda_{\pm}$. The set of singularities of $M(\rho)$ (as an analytic function) coincides with the $\mathbf{R} \cup \Lambda$. For $|\rho| \to \infty$, $\rho \in \Pi^1_{\pm}$,*

$$M(\rho) = \frac{1}{\rho(\beta_1 \mp \omega)}[1]. \tag{23}$$

Theorem 1 follows from (19) and from properties of the functions $\Delta(\rho)$ and $e(0, \rho)$.

Definition 1. The set of singularities of the Weyl-type function $M(\rho)$ is called the spectrum of L (and is denoted by $\sigma(L)$). The values of the parameter ρ, for which equation (1) has nontrivial solutions satisfying (2) and the condtion $y(\infty) = 0$ (i.e. $\lim_{x\to\infty} y(x) = 0$), are called eigenvalues of L, and the corresponding solutions are called eigenfunctions of L.

Thus, $\sigma(L) = \mathbf{R} \cup \Lambda$. The set Λ is the discrete spectrum, and $\mathbf{R}$ is the continuous spectrum. Note that $\mathbf{C} \setminus \sigma(L)$ is the resolvent set of L.

Theorem 2. *L has no eigenvalues for real $\rho \neq 0$.*

Proof. For real $\rho \neq 0$, the functions $e_+(x,\rho)$ and $e_-(x,\rho)$ are solutions of equation (1), and in view of (4),

$$e_\pm(x,\rho) \sim \exp(\pm(i\rho x - Q(x))), \text{ as } x \to \infty. \tag{24}$$

Using (24) and Liouville's formula for the Wronskian we calculate

$$\langle e_+(x,\rho), e_-(x,\rho)\rangle = -2i\rho. \tag{25}$$

Suppose that a real number $\rho_0 \neq 0$ is an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. By virue of (25), the functions $\{e_+(x,\rho_0), e_-(x,\rho_0)\}$ form a fundamental system of solutions for equation (1), and consequently $y_0(x) = C_1 e_+(x,\rho_0) + C_2 e_-(x,\rho)$. As $x \to \infty$, $y_0(x) \sim 0$, we have $e_\pm(x,\rho_0) \sim \exp(\pm(i\rho_0 x - Q(x))$. But this is possible only if $C_1 = C_2 = 0$, i.e. $y_0 \equiv 0$. Theorem 2 is proved. □

Theorem 3. The countable set Λ' coincides with the set $\{\rho_k\}$ of all non-zero eigenvalues of L. For $\rho_k \in \Lambda'$, the functions $e(x,\rho_k)$ and $\varphi(x,\rho_k)$ are eigenfunctions, and

$$e(x,\rho_k) = \gamma_k \varphi(x,\lambda_k),\ \gamma_k \neq 0. \tag{26}$$

For the eigenvalues $\{\rho_k\}$ the asymptotical formula (15) holds.

Proof. Let $\rho_k \in \Lambda'$. Then $U(e(x,\rho_k)) = \Delta(\rho_k) = 0$ and, by virtue of (4), $\lim_{x\to\infty} e(x,\rho_k) = 0$. Thus, $e(x,\rho_k)$ is an eigenfunction, and ρ_k is an eigenvalue. Moreover, it follows from (22) that $\langle \varphi(x,\rho_k), e(x,\rho_k)\rangle = 0$, and consequently (26) is valid.

Conversely, let $\rho_k \in \Pi_+ \cup \Pi_-$ be an eigenvalue, and let $y_k(x)$ be a corresponding eigenfunction. Clearly, $y_k(0) \neq 0$, $U(y_k(x)) = 0$. Then $y_k(x) = \beta_k^0 \varphi(x,\rho_k)$. Since $\lim_{x\to\infty} y_k(x) = 0$, one gets $y_k(x) = \beta_k^1 e(x,\rho_k)$. This yields (26). Consequently, $\Lambda(\rho_k) = U(e(x,\rho_k)) = 0$, and $\varphi(x,\rho_k)$ and $e(x,\rho_k)$ are eigenfunctions. □

Remark 1. We note that $(\Lambda''_+ \setminus \{0\}) \cap (\Lambda''_- \setminus \{0\}) = \varnothing$, i.e. for real $\rho \neq 0$ the functions $\Delta_+(\rho)$ and $\Delta_-(\rho)$ are not equal to zero simultaneously. Indeed, it follows from (8) and (25) that for real $\rho \neq 0$, one has

$$0 \neq \langle e_+(x,\rho), e_-(x,\rho)\rangle = e_+(0,\rho)e'_-(0,\rho) - e'_+(0,\rho)e_-(0,\rho) =$$
$$= e_+(0,\rho)\Delta_-(\rho) - e_-(0,\rho)\Delta_+(\rho).$$

For brevity, we confine ourselves to the case of a simple spectrum in the following sense.

Definition 2. We shall say that L has simple spectrum if all zeros of $\Delta(\rho)$ are simple, have no finite limit points, and $\rho M(\rho) = m_\pm + o(1)$ as $\rho \to \infty$ FORMYLA, $m_\pm \in \mathbf{C}$.

Let L have simple spectrum. Then Λ'' is a finite set, and $\Lambda = \Lambda' \cup \Lambda''$ is a countable set:

$$\Lambda = \{\rho_k\}_{k\in\omega}.$$

Here $\omega = \omega_0 \cup \omega^0$, where ω_0 is a finite set, $\omega_0 = \{k \in \mathbf{Z} : |k| >_0 k\}$ for some k_0, and the numbers ρ_k have the form (15) for $k \in \omega^0$. Each element of Λ' is an eigenvalue of L. According to Theorem 2, the points of $\Lambda''\setminus\{0\}$ are not eigenvalues of L, they are called *spectral singularities* of L. Thus, the discrete spectrum Λ consists of two parts: the set of eigenvalues, and the set of spectral singularities.

Denote

$$M_k = \frac{e(0,\rho_k)}{\dot\Delta(\rho_k)},\ \rho_k \in \Lambda\setminus\{0\}. \tag{27}$$

Obviously, $M_k \neq 0$, and

$$\lim_{\rho\to\rho_k,\rho\in\overline{\Pi}_\pm}(\rho-\rho_k)M(\rho)=M_k,\ \rho_k\in\Lambda_\pm\setminus\{0\}. \tag{28}$$

Let

$$\alpha_k:=\begin{cases}M_k & \text{for } \rho_k\in\Lambda',\\ \frac{1}{2}M_k & \text{for } \rho_k\in\Lambda''\setminus\{0\},\end{cases} \tag{29}$$

$$V(\rho)=\frac{1}{2\pi i}(M^-(\rho)-M^+(\rho)),\rho\in\Pi:=\mathbf{R}\setminus\Lambda''. \tag{30}$$

where $M^\pm(\rho):=\lim_{z\to 0,z\in\Pi_\pm}M(\rho\pm iz)=\frac{e_\pm(0,\rho)}{\Delta_+(\rho)}$. Put $\alpha_0=(m_++m_-)/(\pi i)$ for $\rho_0=0$. Using (13), (14), (15), (27) and (29) we calculate

$$\alpha_k=\mp\frac{\omega}{k\pi a(\beta_1^2-\omega^2)}+O(\frac{1}{k^2}),\ \ k\to\pm\infty. \tag{31}$$

By virtue of (19) and (30),

$$V(\rho)=\frac{1}{2\pi i}\left(\frac{e_-(0,\rho)}{\Delta_-(\rho)}-\frac{e_+(0,\rho)}{\Delta_+(\rho)}\right),\ \rho\in\Pi.$$

Taking (8) and (25) into account we infer

$$V(\rho)=\frac{\rho}{\pi}\cdot\frac{1}{\Delta_-(\rho)\Delta_+(\rho)},\ \rho\in\Pi. \tag{32}$$

Definition 3. The data $S:=(\{V(\rho)\}_{\rho\in\Pi},\{\rho_k,\alpha_k\}_{k\in\omega})$, are called the *spectral data* of L.

The spectral data describe the behavior of the spectrum; $\{V(\rho)\}$ is connected with the continuous spectrum, and $\{\lambda_k,\alpha_k\}_{k\in\omega}$ describe the discrete spectrum. Using the results obtained above we arrive at the following statement.

Theorem 4. *The spectral data* $S:=(\{V(\rho)\}_{\rho\in\Pi},\{\rho_k,\alpha_k\}_{k\in\omega})$ *have the following properties:*

(i_1) $\rho_k\neq\rho_s$, *for* $k\neq s$; *moreover,* $(\Lambda''_+\setminus\{0\})\cap(\Lambda''_-\setminus\{0\})=\varnothing$;

(i_2) *as* $k\to\pm\infty$, *the asymptotical formulas (15) and (31) are valid;*

(i_3) the function $V(\rho)$ is continuously differentable for $\rho\in\Pi$, and for $\rho_k\in\Lambda''$ there exist finite limits $V_k=\lim_{\rho\to\rho_k}(\rho-\rho_k)V(\rho)$; moreover,

$$V_k=\mp\frac{\alpha_k}{\pi i}\text{ for }\rho_k\in\Lambda''_\pm\setminus\{0\}; \tag{33}$$

(i_4) *as* $\rho\to\pm\infty$,

$$V(\rho)=\frac{4[1]}{\pi\rho(\beta_1\mp\omega)^2(1+1/\omega)^2}\exp(\pm(2\omega\rho a-2iQ/\omega)). \tag{34}$$

The asymptotics (34) follows from (32) and (9). Notice that relation (33) gives us a connection between $V(\rho)$, which describes the continuous spectrum, and $\{\rho_k,\alpha_k\}$, which describe the discrete spectrum.

3. Formulation of the inverse problem. The uniqueness theorem

Let us go on to studying the inverse problem for the boundary value problem L. The inverse problem is formulated as follows.

Inverse Problem 1. *Given the spectral data S, construct the coefficients of the pencil (1)-(2).*

In this section we prove the uniqueness theorem for the solution of this inverse problem. For this purpose together with L we will consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{r}(x)$, $\tilde{p}(x)$, $\tilde{q}(x)$, $\tilde{\beta}_1$, $\tilde{\beta}_0$. We agree that if a certain symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$ and $\hat{\alpha} = \alpha - \tilde{\alpha}$.

Theorem 5. *If $S = \tilde{S}$, then $r(x) = \tilde{r}(x), p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ for $x > 0$, $\beta_1 = \tilde{\beta}_1$ and $\beta_0 = \tilde{\beta}_0$. Thus, the specification of the spectral data uniquely determines the coefficients of the pencil (1)-(2).*

Proof. Fix $\delta > 0$. Let $\kappa_\delta^0(\rho_k) := \{\rho : \rho \in [\rho_k - \delta, \rho_k + \delta]\}$, $\rho_k \in \Lambda''$. Denote by $\xi_\delta := \mathbf{R} \setminus \left(\bigcup_{\rho_k \in \Lambda''} \kappa_\delta^0(\rho_k)\right)$ the real axis without δ- neighbourhoods of the points of Λ''.

Let us show that the specification of the spectral data S uniquely determines the Weyl-type function $M(\rho)$ via the formula

$$M(\rho) = \sum_{\rho_k \in \Lambda} \frac{\alpha_k}{\rho - \rho_k} + \int_{-\infty}^{+\infty} \frac{V(\mu)}{\rho - \mu} d\mu, \quad \rho \notin \sigma(L), \tag{35}$$

where the integral is understood in the principal value sense: $\int_{-\infty}^{+\infty} := \lim_{\delta \to 0} \int_{\xi_\delta}$.

Indeed, fix $\delta > 0$ and denote $G_\delta := \{\rho \in \mathbf{C} : |\rho - \rho_k| \geq \delta, \rho_k \in \Lambda\}$. It follows from (9), (13) and (19) that

$$|\Delta(\rho)| \geq C |\rho| \exp(|\sigma| \omega a) \exp(-|\tau| a), \; |M(\rho)| \leq C|\rho|^{-1}, \; \rho \in G_\sigma, \tag{36}$$

where $\sigma := \operatorname{Re} \rho$, $\tau := \operatorname{Im} \rho$, i.e. $\rho = \sigma + i\tau$. According to (34), the integral in (35) converges absolutely at infinity. Moreover, in view of (15) and (31), the series in (35) converges absolutely too.

Take positive numbers $R_N = \dfrac{N\pi}{\omega a} + \chi$ such that the circles $\theta_N := \{\rho : |\rho| = R_N\}$ lie in G_δ for sufficiently small $\delta > 0$. Fix $\rho \notin \sigma(L)$, and take $\delta > 0$ and N such that $\rho \in G_\delta \cap \operatorname{int} \theta_N$. Consider the contour integral

$$I_N(\rho) := \frac{1}{2\pi i} \int_{\theta_N} \frac{M(\mu)}{\rho - \mu} d\mu \tag{37}$$

with counterclockwise circuit. It follows from (36) that

$$\lim_{N \to \infty} I_N(\rho) = 0. \tag{38}$$

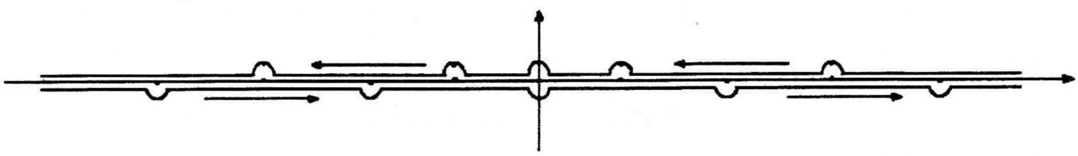

fig.1

For each $\rho \in \Lambda_\pm''$ we take a semicircle $\kappa_\delta(\rho_k) := \{\rho : |\rho - \rho_k| = \delta, \rho \in \Pi_\pm\}$. Let Π_δ be the two-sided cut Π_0 without the δ- neighbourhoods of the points of Λ'', and let $\Gamma_\delta := \Pi_\delta \cup \left(\bigcup_{\rho_k \in \Lambda''} \kappa_\delta(\rho_k) \right)$ be the contour with counterclockwise circuit (see fig.1). Denote $\Gamma_{\delta,N} := \Gamma_\delta \cap \theta_{N,0}$, where $\theta_{N,0} = \{\rho : |\rho| \le R_N\}$. Contracting the contour in (37) to the real axis through the poles of Λ' and using (19), (27), (29) and the residue theorem, we get

$$M(\rho) = \sum_{\substack{\rho_k \in \Lambda' \\ |\rho_k| < R_N}} \frac{\alpha_k}{\rho - \rho_k} + \frac{1}{2\pi i} \int_{\Gamma_{\delta,N}} \frac{M(\mu)}{\rho - \mu} d\mu - I_N(\rho).$$

By virtue of (38) this yields as $N \to \infty$:

$$M(\rho) = \sum_{\rho_k \in \Lambda'} \frac{\alpha_k}{\rho - \rho_k} + \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{M(\mu)}{\rho - \mu} d\mu. \tag{39}$$

Taking (28)-(30) into account we calculate

$$\lim_{\delta \to 0} \sum_{\rho_k \in \Lambda'} \frac{1}{2\pi i} \int_{\kappa_\delta(\rho_k)} \frac{M(\mu)}{\rho \mp \mu} d\mu = \sum_{\rho_k \in} \frac{\alpha_k}{\rho - \rho_k}, \quad \frac{1}{2\pi i} \int_{\Pi_\delta} \frac{M(\mu)}{\rho - \mu} d\mu = \int_{\xi_\delta} \frac{V(\mu)}{\rho - \mu} d\mu.$$

Therefore, from (39) as $\delta \to \infty$ we arrive at (35).

Furthermore, it follows from (6) and (12) that for $\rho \in \overline{\Pi_\pm}$, $m = 0,1$,

$$\left. \begin{aligned} &|e^{(m)}(x,\rho)| \le C|\rho|^m \exp(|\sigma|\omega(a-x))\exp(-|\tau|a), \quad \text{for} \quad x \le a, \\ &|e^{(m)}(x,\rho)| \le C|\rho|^m \exp(-|\tau|a), \quad \text{for} \quad x \ge a. \end{aligned} \right\} \tag{40}$$

Using (17), (36) and (40) we conclude that for $\rho \in G_\sigma$, $m = 0,1$,

$$\left. \begin{aligned} &|\Phi^{(m)}(x,\rho)| \le C|\rho|^{m-1} \exp(-|\sigma|\omega x), \quad \text{for} \quad x \le a, \\ &|\Phi^{(m)}(x,\rho)| \le C|\rho|^{m-1} \exp(-|\sigma|\omega a)\exp(-|\tau|(x-a)), \quad \text{for} \quad x \ge a. \end{aligned} \right\} \tag{41}$$

Now we need to study the asymptotic behavior of the solution $\varphi(x,\rho)$ as $|\rho| \to \infty$. Using the Birkhoff-type fundamental system of solutions $\{y_k(x,\rho)\}_{k=1,2}$ of equation (1) on the interval $[0, a]$, one has

$$\varphi(x,\rho) = a_1(\rho)y_1(x,\rho) + a_2(\rho)y_2(x,\rho), x \in [0,a]. \tag{42}$$

Let $\{Y_k(x,\rho)\}_{k=1,2}$, $x \ge a$, $\rho \in \overline{\Pi_\pm}$ be the Birkhoff-type fundamental system of solutions of equation (1) on the interval $[a, \infty)$, with the asymptotics for $\rho \to \infty$, $m = 0,1$.

$$Y_k^{(m)}(x,\rho) = ((-1)^{k-1} i\rho)^m \exp((-1)^k \rho(i \ \ x - iQ(x)))[1]; \tag{43}$$

(see [1], [28]-[29]). Then

$$\varphi(x,\rho) = A_1(\rho)Y_1(x,\rho) + A_2(\rho)Y_2(x,\rho), x \ge a. \tag{44}$$

Taking (10) and the initial conditions $\varphi(0,\rho) = 1$, $\varphi'(0,\rho) = \beta_1\rho + \beta_0$ into account, we calculate

$$a_1(\rho)[1] + a_2(\rho)[1] = 1, \quad (\beta_1 - \omega)a_1(\rho)[1] + (\beta_1 + \omega)a_2(\rho)[1] = 0,$$

and consequently,

$$a_1(\rho)[1] = \frac{\omega+\beta_1}{2\omega}[1],\ a_2(\rho)[1] = \frac{\omega-\beta_1}{2\omega}[1],\ \rho \to \infty. \tag{45}$$

Substituting (10) and (45) into (42) we obtain the asymptotical formula for $\varphi^{(m)}(x, \rho)$, $m = 0,1$ as $\rho \to \infty$, uniformly in $x \in [0, a]$:

$$\varphi^{(m)}(x,\rho) = \frac{1}{2\omega}((-\omega\rho)^m (\omega+\beta_1)\exp(-\omega\rho x + iQ(x)/\omega)[1] +$$
$$+(-\omega\rho)^m(\omega-\beta_1)\exp(\omega\rho x - iQ(x)/\omega)[1]). \tag{46}$$

In order to calculate the coefficients $A_k(\rho)$, k = 1,2, we use (43), (44), (46) and the smooth conditions $\varphi^{(m)}(a-0,\rho) = \varphi^{(m)}(a+0,\rho)$, $m = 0,1$. This yields for $\rho \to \infty$:

$$A_1(\rho)\exp(i\rho a - Q)[1] + (-1)^m A(\rho)\exp(-i\rho a + Q)[1] = (i\rho)^- \varphi^{(m)}(a,\rho),\ m=0,1,$$

where the asymptotics for $\varphi^{(m)}(a, \rho)$ is taken from (46). Calculating $A_k(\rho)$ from this algebraic system and substituting the result and (43) into (44) we get for $x \geq a$, $\rho \to \infty$:

$$\varphi^{(m)}(x,\rho) = \frac{1}{4}\exp(i\rho(x-a) - Q_a(x))((\omega+\beta_1)(1/\omega+i)\exp(-\omega\rho a + iQ/\omega)[1] +$$
$$+(\omega-\beta_1)(1/\omega-i)\exp(\omega\rho a - iQ/\omega)[1]) +$$
$$+\frac{1}{4}\exp(-i\rho(x-a) + Q_a(x))((\omega+\beta_1)(1/\omega-i)\exp(-\omega\rho a + iQ/\omega)[1] +$$
$$+(\omega-\beta_1)(1/\omega+i)\exp(\omega\rho a - iQ/\omega)[1]),\ Q_a(x) := \frac{1}{2}\int_a^x q_1(t)dt. \tag{47}$$

It follows from (46) and (47) that

$$\left.\begin{aligned} &|\varphi^{(m)}(x,\rho)| \leq C|\rho|^m \exp(|\sigma|\omega x), \quad \text{for } \leq x \quad a, \\ &|\varphi^{(m)}(x,\rho)| \leq C|\rho|^m \exp(|\sigma|\omega a)\exp(|\tau|(x \quad a)), \quad \text{for} \quad x \geq a. \end{aligned}\right\} \tag{48}$$

By the assumption of Theorem 5, $S = \tilde{S}$. Hence, in view of (35),

$$M(\rho) \equiv \tilde{M}(\rho). \tag{49}$$

Using (15), (23), (31) and (49) we in $M(\rho) \equiv \tilde{M}(\rho)$ fer

$$\beta_1 = \tilde{\beta}_1,\ \omega = \tilde{\omega},\ a = \tilde{a},\ Q = \tilde{Q}. \tag{50}$$

Let us now define the matrix $P(x,\rho) = [P_{j,k}(x,\rho)]_{j,k=1,2}$ by the formula

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\Phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\Phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \Phi(x,\rho) \\ \varphi'(x,\rho) & \Phi'(x,\rho) \end{bmatrix}. \tag{51}$$

Note that the idea of applying mappings of the solution space of differential equations for studying the inverse problem is due to Leibenzon [31]. By virtue of (21) this yields

$$\left.\begin{aligned} P_{j1}(x,\rho) &= \varphi^{(j-1)}(x,\rho)\tilde{\Phi}'(x,\rho) - \Phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) &= \Phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\Phi}(x,\rho), \end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned}\varphi(x,\rho)=P_{11}(x,\rho)\tilde\varphi(x,\rho)+P_{12}(x,\rho)\tilde\varphi'(x,\rho),\\ \Phi(x,\rho)=P_{21}(x,\rho)\tilde\Phi(x,\rho)+P_{22}(x,\rho)\tilde\Phi'(x,\rho).\end{aligned}\right\} \tag{53}$$

It follows from (41), (48) and (52) that for $x\ge a$, $\rho\in G_\sigma$,

$$|P_{11}(x,\rho)|\le C,\quad |P_{12}(x,\rho)|\le C|\rho|^{-1}. \tag{54}$$

Using (20) and (52) we calculate

$$P_{j1}(x,\rho)=\varphi^{(j-1)}(x,\rho)\tilde S'(x,\rho)-S^{(j-1)}(x,\rho)\tilde\varphi'(x,\rho)+(\tilde M(\rho)-$$
$$-M(\rho))\varphi^{(j-1)}(x,\rho)\tilde\varphi'(x,\rho),$$
$$P_{j2}(x,\rho)=S^{(j-1)}(x,\rho)\tilde\varphi(x,\rho)-\varphi^{(j-1)}(x,\rho)\tilde S(x,\rho)+$$
$$+(M(\rho)-\tilde M(\rho))\varphi^{(j-1)}(x,\rho)\tilde\varphi(x,\rho).$$

Taking (49) into account we conclude that the functions $P_{jk}(x,\rho)$ are entire in ρ for each fixed $x\ge 0$. Together with (54) this yields $P_{12}(x,\rho)\equiv 0$, $P_{11}(x,\rho)\equiv P_1(x)$, i.e. the function P_{11} does not depend on ρ. By virtue of (53) we have for all x and ρ:

$$P_1(x)\tilde\varphi(x,\rho)\equiv\varphi(x,\rho),\quad P_1(x)\tilde\Phi(x,\rho)\equiv\Phi(x,\rho). \tag{55}$$

Let $x\in[0,a]$. Using (9), (12), (17), (46) and (50) we get as $\rho\to\infty$, $\arg\rho\in(0,\pi/2)$:

$$\frac{\varphi(x,\rho)}{\tilde\varphi(x,\rho)}=\exp(-i(Q(x)-\hat Q(x))/\omega)[1],\quad \frac{\Phi(x,\rho)}{\tilde\Phi(x,\rho)}=\exp(i(Q(x)+\hat Q(x))/\omega)[1]. \tag{56}$$

Since $P_1(x)$ does not depend on ρ, it follows from (55) and (56) that

$$P_1(x)\equiv\exp(-i(Q(x)-\tilde Q(x))/\omega)[1],\ P_1(x)\equiv\exp(i(Q(x)+\tilde Q(x))/\omega)[1].$$

and consequently, $Q(x)\equiv\tilde Q(x)$, $P_1(x)\equiv 1$ for $x\in[0,a]$.

Let $x\ge 0$. Using (6), (9), (17), (47) and (50) we have as $\rho\to\infty$, $\arg\rho\in(0,\pi/2)$:

$$\frac{\varphi(x,\rho)}{\tilde\varphi(x,\rho)}=\exp(\hat Q_a(x)-i\hat Q)[1],\quad \frac{\Phi(x,\rho)}{\tilde\Phi(x,\rho)}=\exp(-(\hat Q_a(x)-i\hat Q))[1]. \tag{57}$$

Since $P_1(x)$ does not depend on ρ and $Q=\tilde Q$., it follows from (55) and (57) that

$$P_1(x)\equiv\exp(\hat Q_a(x)),\quad P_1(x)\equiv\exp(-\hat Q_a(x)),$$

and consequently, $Q_a(x)\equiv\tilde Q_a(x)$, $P_1(x)\equiv 1$ for $x\ge 0$. Thus, $P_1(x)\equiv 1$ and $q_1(x)\equiv\tilde q_1(x)$ for all $x\ge 0$. According to (55) this yields $\tilde\varphi(x,\rho)\equiv\varphi(x,\rho)$, $\tilde\Phi(x,\rho)\equiv\Phi(x,\rho)$. Hence, $q_0(x)\equiv\tilde q_0(x)$ a.e. on $(0,\infty)$ and $\beta_0=\tilde\beta_0$. Theorem 5 is proved. □

Corollary 1. *If $M(\rho)\equiv\tilde M(\rho)$, then $L=\tilde L$.*

It follows from the proof of Theorem 5 that the last assertion is also valid for pencil (1)-(2) with arbitrary behavior of the spectrum.

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References

1. *Tamarkin J.D.* On some problems of the theory of ordinary linear dif ferential equations. Petrograd, 1917.

2. *Keldysh M.V.* On eigenvalues and eigenfunctions of some classes of nonselfadjoint equations // Dokl. Akad. Nauk SSSR. 1951. V. 77. P. 11–14.

3. *McHugh J.* An historical survey of ordinary linear differential equations with a large parameter and turning points // Arch. Hist. Exact. Sci. 1970. V. 7. P 277–324.

4. *Kostyuchenko A.G., Shkalikov A.A.* Selfadjoint quadratic operator pencils and elliptic problems // Funkt. Anal. Prilozhen. 1983. V. 17, N 2. P. 38–61 (Russian); Funct. Anal. Appl. 1983. V. 17, N 2. P. 109–128 (English).

5. *Freiling G.* On the completeness and minimality of the derived chains of eigen and associated functions of boundary eigenvalue problems nonlinearly dependent on the parameter // Results in Math. 1988. V. 14. P. 64–83.

6. *Wasow W.* Linear turning point theory. Berlin, 1985.

7. *Eberhard W., Freiling G.* An expansion theorem for eigenvalue problems with several turning points // Analysis. 1993. V. 13. P. 301–308.

8. *Beals R.* Indefinite Sturm-Liouville problems and half-range completeness // J. Diff. Equations. 1985. V. 56, N 3. P. 391–407.

9. *Langer H., Curgus B.* A Krein space approach to symmetric ordinary differential operators with an indefinite weight function // J. Diff. Equations. 1989. V. 79, N 1. P. 31–61.

10. *Marchenko V.A.* Sturm-Liouville operators and their applications. Kiev, 1977 (Russian); Birkhauser, 1986.

11. *Levitan B.M.* Inverse Sturm-Liouville problems. M., 1984 (Russian); Utrecht, 1987.

12. *Freiling G., Yurko V.A.* Inverse Sturm-Liouville problems and their applications. N. Y., 2001.

13. *Gasymov M.G., Gusejnov G.S.* Determination of diffusion operators according to spectral data // Dokl. Akad. Nauk Az. SSR. 1981. V. 37, N 2. P. 19–23.

14. *Yamamoto M.* Inverse eigenvalue problem for a vibration of a string with viscous drag // J. Math. Anal. Appl. 1990. V/ 152, N 1. P. 20–34.

15. *Khruslov E.Y., Shepelsky D.G.* Inverse scattering method in electromagnetic sounding theory // Inverse Problems. 1994. V. 10, N 1. P. 1–37.

16. *Yurko V.A.* An inverse problem for systems of differential equations with nonlinear dependence on the spectral parameter // Diff. Uravneniya. 1997. V. 33, N 3. P. 390–395 (Russian); Diff. Equations. 1997. V. 33, N 3. P. 388–394 (English).

17. *Aktosun T., Klaus M., Mee C. van der.* Inverse scattering in one-dimensional nonconservative media // Integral Equat. Oper. Theory. 1998. V. 30, N 3. P. 279–316.

18. *Pivovarchik V.* Reconstruction of the potential of the Sturm-Liouville equation from three spectra of boundary value problems // Funct. Anal. i Prilozh. 1999. V. 33, N 3. P. 87–90 (Russian); Funct. Anal. Appl. 1999. V. 33, N 3. P. 233–235 (English).

19. *Yurko V.A.* An inverse problem for pencils of differential operators // Mat. Sbornik. 2000. V. 191, N 10. P. 137–160 (Russian); Mathematics. 2000. V. 191, N 10. P. 1561–1586 (English).

20. *Belishev M.I.* An inverse spectral indefinite problem for the equation $y»+\lambda r(x)y=0$ on an interval // Funct. Anal. i Prilozh. 1987. V. 21, N 2. P. 68–69 (Russian); Funct. Anal. Appl. 1987. V. 21, N 2. P. 146–148 (English).

21. *Darwish A.A.* On the inverse scattering problem for a generalized Sturm-Liouville differential operator // Kyungpook Math. J. 1989. V. 29, N 1. P. 87–103.

22. *El-Reheem, Zaki F.A.* The inverse scattering problem for some singular Sturm-Liouville operator // Pure Math. Appl. 1997. V. 8, N 2–4. P. 233–246.

23. *Freiling G., Yurko V.A.* Inverse problems for differential equations with turning points // Inverse Problems. 1997. V. 13. P. 1247–1263.

24. *Freiling G., Yurko V.A.* Inverse spectral problems for differential equations on the half-line with turning points // J. Diff. Equations. 1999. V. 154. P. 419–453.

25. *Bennewitz C.* A Paley-Wiener theorem with applications to inverse spectral theory. Advances in diff. equations and math. physics. Birmingham, AL, 2002. P. 21–31; Contemp. Math. Amer. Math. Soc. Providence, RI, 2003. V. 327.

26. *Yurko V.A.* Method of spectral mappings in the inverse problem theory: Inverse and ill-posed problems series. Utrecht, 2002.

27. *Coddington E., Levinson N.* Theory of ordinary differential equations. N. Y., 1955.

28. *Rykhlov V.S.* Asymptotical formulas for solutions of linear differential systems of the first order // Results Math. 1999. V. 36, N 3-4. P. 342–353.

29. *Mennicken R., Moeller M.* Non-self-adjoint boundary eigenvalue problems. Amsterdam, 2003.

30. *Levitan B.M., Sargsjan I.S.* Introduction to spectral theory. M., 1970 (Russian); AMS Transl. of Math. Monographs. Providence, RI, 1975. V. 39 (English).

31. *Leibenzon Z.L.* The inverse problem of spectral analysis for higher-order ordinary differential operators. // Trudy Mosk. Mat. Obshch. 1966. V. 15. P. 70–144 (Russian); Trans. Moscow Math. Soc. 1966. V. 15 (English).