Symmetrization in Clean and Nil-Clean Rings

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We introduce and investigate *D-clean* and *D-nil-clean rings* as well as some other closely related symmetric versions of cleanness and nil-cleanness. A comprehensive structural characterization is given for these symmetrically clean and symmetrically nil-clean rings in terms of Jacobson radical and its quotient. It is proved that strongly clean (resp., strongly nil-clean) rings are always D-clean (resp., D-nil-clean).Our results corroborate our recent findings published in Bull. Irkutsk State Univ., Math. (2019) and Turk. J. Math. (2019). We also show that weakly nil-clean rings defined as in Danchev-McGovern (J. Algebra, 2015) and Breaz – Danchev – Zhou (J. Algebra & Appl., 2016) are actually weakly nil clean in the sense of Danchev-Šter (Taiwanese J. Math., 2015). This answers the question of the reviewer D. Khurana (Math. Review, 2017).

Keywords: L-clean rings, R-clean rings, D-clean rings, symmetrization, weakly nil(-)clean rings.

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1. Introduction and Background

Throughout this paper, all rings R are assumed to be associative and unital with identity element 1 different from the zero element 0 of R. Our standard terminology and notations are mainly in agreement with [1]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ denotes the set of all idempotents in R, $Nil(R)$ denotes the set of all nilpotents in R, and $J(R)$ denotes the Jacobson radical of R. The specific notions will be provided in the present article.

The fundamental paper [2] introduced and studied the class of *clean rings* R as those rings for which $R = U(R) + Id(R)$. Correspondingly, a *nil-clean ring* R is one for which $R = Nil(R) + Id(R)$ (see [3] and [4]). It is pretty easy to check that nil-clean rings are always clean, but not the converse.

In making up an attempt to simplify these two too complicated ring structures, in [5] we defined both L-clean and R-clean rings (resp., L-nil-clean and R-nil-clean rings) as members of subclasses of the classes of clean and nil-clean rings, respectively, possessing left or right symmetric property of the existing idempotent elements. For instance, a ring R is called *L-clean* if, for any $a \in R$, there is $e \in aR \cap Id(R)$ with $a = (1 - e) + u$ for some $u \in U(R)$ and, resp., it is called *R-clean* if, for any $a \in R$, there is $e \in Ra \cap Id(R)$ with $a = (1 - e) + u$ for some $u \in U(R)$. Analogous way of defining *L-nil-clean* and *R-nil-clean* leads to $q \in Nil(R)$ with $a = e + q$, where either $e \in aR \cap Id(R)$ or $e \in Ra \cap Id(R)$. It was proved there that L-nil-clean (resp., R-nil-clean) rings are L-clean (resp., R-clean).

This provides us with our first basic tool.

Definition 1. A ring R is said to be *double nil-clean* or just *D-nil-clean* for short if, for every $a \in R$, there exists $e \in (aRa) \cap Id(R)$ such that $a = e + q$ for some $q \in Nil(R)$.

Certainly, D-nil-clean rings are nil-clean. As an immediate example, Boolean rings are clearly of a such kind. Besides, the condition $e \in aRa$ is obviously equivalent to $e \in aR \cap Ra$ as $aRa \subseteq aR \cap Ra$ and if $e = ab = ca$ for some $b, c \in R$ then it follows that $e = e.e = abca \in aRa$ as needed. This illustrates that D-nil-clean rings are both L-nil-clean and R-nil-clean; however, the converse is still unknown.

Another, non-commutative, example is the ring $R = \mathbb{M}_2(\mathbb{Z}_2)$. In fact, all elements of this matrix ring are nilpotents, idempotents and nil-clean units. If $q \in R$ is a nilpotent, then $q = 0 + q$ and $0 \in qRq$. Next, if $e \in R$ is an idempotent, then $e = e + 0$ and $e \in eRe$ because $e = e.1.e$ or $e = e.e.e$. If $u \in R$ is a unit with $u = e + q$ for some idempotent $e \in R$ and a nilpotent $q \in R$, then $e = u - q = u(u^{-1} - u^{-1}qu^{-1})u \in uRu$, as we need. Q.E.D.

This provides us with our second basic tool.

Definition 2. A ring R is said to be *double clean* or just *D-clean* for short if, for each $a \in R$, there exists $e \in (aRa) \cap Id(R)$ such that $a = (1 - e) + u$ for some $u \in U(R)$.

Certainly, D-clean rings are clean. As an example of a D-clean ring, strongly regular rings are clearly of a such kind. As above demonstrated, D-clean rings are both L-clean and R-clean; however, the converse is still unknown. What may also be observed is that D-nil-clean rings are themselves D-clean. Indeed, for any $a \in R$, in view of Definition 1 $a = e + q$ for some $e \in Id(R) \cap (aRa)$ and $q \in Nil(R)$. Therefore $a = (1 - e) + (2e + q - 1)$. Since $2 \in Nil(R)$ (see, for instance, [3]) by simple operations — omitting some details — we find that $2e + q \in Nil(R)$ and $2e + q - 1 \in U(R)$. It meets our needs.

However, in [6] we defined the two concepts of *regularly nil clean rings* and *Utumi rings* as follows: a ring R is regularly nil clean if, for every $a \in R$, there is $e \in Ra \cap Id(R)$ such that $a(1 - e) \in Nil(R)$ and $(1 - e)a \in Nil(R)$ or, in an equivalent form, there is $f \in aR \cap Id(R)$ such that $a(1 - f) \in Nil(R)$ and $(1 - f)a \in Nil(R)$. It was also shown in [6, Proposition 2.5] that all regularly nil clean rings are Utumi rings in the sense that, for each $x \in R$, there is $y \in R$ depending on x such that $x - x^2y \in Nil(R)$.

So, analyzing all the above, the aim of the present article is to develop a theory of double cleanness, nil-cleanness and regular nil cleanness as well as some their modifications caused by the symmetry of idempotents. Specifically, we will give a satisfactory (complete) description of these three classes of rings. We will also settle a recent question in [6, p. 703], on whether or not Utumi rings are somewhat left-right symmetric in the sense that $x - yx^2 \in Nil(R)$.

The work is structured as follows: the next section states and proves our major results (see, respectively, Propositions 1, 2 and 3, Lemma 1, as well as Remark 1 listed below): the final part consists of some useful commentaries on the more insightful exploration of the current subject and a list of problems that remain open.

2. Preliminary and Main Results: Symmetrically Clean and Nil-Clean Rings

We begin here with our first main result concerning a symmetrization of Utumi rings (actually, this was stated as a problem that remains open in [6, p. 703]).

Proposition 1. *The rings of Utumi are left-right symmetric.*

Proof. Let $x \in R$ be an arbitrary element. Hence, by definition, there is $y \in R$ depending on x such that $x - x^2y \in Nil(R)$. We claim that $x - yx^2 \in Nil(R)$ shows the desired symmetry. In fact, for all $n \in \mathbb{N}$, $(x - x^2y)^n = [x(1 - xy)]^n = x(x - xyx)^{n-1}(1 - xy)$. Thus, if $(x - x^2y)^n = 0$, then one observes that $(x - xyx)^{n+1} = (1 - xy)(x - x^2y)^n x = 0$.

Since analogously $(x-yx^2)^{n+2} = (1-yx)(x-xyx)^{n+1}x$, we see that this is zero as well, i.e., $(x-yx^2)^{n+2} = 0$. □

The following lemma is useful for our investigation.

Lemma 1. *Suppose that R is a ring. Then the following items are valid:*
(1) R is D-nil-clean if and only if $R/J(R)$ is D-nil-clean and $J(R)$ is nil;
(2) R is D-clean if and only if $R/J(R)$ is D-clean, provided that $J(R)$ is nil.

Proof. Before proving the two statements separately, we need the following fact:

If K is a ring with a nil-ideal I and if $d \in K$ with $d+I \in Id(K/I)$, then $d+I = e+I$ for some $e \in Id(K) \cap dKd$ such that $de = ed$.

(1) The left-to-right implication being elementary, let us focus on the right-to-left one. So, given $r \in R$, one writes by assumption that $r + J(R) = (e + J(R)) + (q + J(R))$ for some $e, q \in R$ such that $e + J(R)$ is an idempotent in $R/J(R)$ having the property $e + J(R) \in (a + J(R))(R/J(R))(a + J(R))$, and $q + J(R)$ is a nilpotent in $R/J(R)$. As $J(R)$ is nil, one easily understands that q has to be a nilpotent as well. As for the element e, there is $c \in R$ such that $e + J(R) = aca + J(R)$. In view of the aforementioned fact, $e + J(R) = f + J(R)$ for some idempotent f of R possessing the property that $f \in (aca)R(aca) \subseteq aRa$. Thus, $e \in f + J(R)$ and, finally, $r \in f + Nil(R)$, because $q + J(R) \subseteq Nil(R)$.

(2) The left-to-right implication being trivial, let us concentrate on the right-to-left one. So, given $r \in R$, one writes that $e + J(R)$ is an idempotent in $R/J(R)$ possessing the property $e+J(R) \in (a+J(R))(R/J(R))(a+J(R))$, and $u+J(R)$ is a unit in $R/J(R)$. Since the containment $1 + J(R) \subseteq U(R)$ holds, it is obvious that $u \in U(R)$. The rest of the proof is hereafter identical to that in point (1). Q.E.D. □

Let us recall that a ring is termed *strongly nil-clean* if its elements are sums of a nilpotent and an idempotent which commute.

The next implication is of interest.

Proposition 2. *Strongly nil-clean rings are D-nil-clean.*

Proof. For an arbitrary element r of such a ring R, such that $r = q + e$ for some $q \in Nil(R)$ and $e \in Id(R)$ with $qe = eq$, it follows that $r - e = q$ with $re = er$. Thus $(r-e)^k = 0$ for some $k \in \mathbb{N}$ and, expanding this by the classical binomial formula, one derives by a direct inspection that $e \in rR \cap Rr$. It leads to $e = e.e \in rRr$. Q.E.D. □

Let us notice that an alternative proof could also be deduced by using the fact from [7, Theorem B] that a ring R is strongly nil-clean if, and only if, the factor-ring $R/J(R)$ is boolean and the ideal $J(R)$ is nil, as stated in Lemma 1 (1) and the simple but useful observation that boolean rings are always D-nil-clean being commutative rings containing only idempotents.

Let us recall that a ring R is termed *strongly π-regular* if, for each $a \in R$, there is $n \in \mathbb{N}$ depending on a having the property $a^n \in a^{n+1}R \cap Ra^{n+1}$ (see, e.g., [8]). It is well known that strongly nil-clean rings are always strongly π-regular rings, whereas strongly π-regular rings are always *strongly clean* rings (see, e.g., [9]) in the sense that their elements are sums of a unit and an idempotent that commute.

Proposition 3. *Strongly clean rings (and, in particular, strongly π-regular rings) are D-clean.*

Proof. For an arbitrary element r of such a ring R, we may write that $1-r = u+e$ for some $u \in U(R)$ and $e \in Id(R)$ with $ue = eu$. Hence $r = (-u) + (1-e)$ with $ru = ur$

and $re = er$, so that $re = (-u)e$ with $e \in (-u)^{-1}re = r(-u)^{-1}e = (-u)^{-1}er \in rR \cap Rr$. By a direct inspection, one deduces that $e = e.e \in rRr$. Q.E.D. □

We will be further concerned with the clarifications of two concepts concerning weakly nil(-)clean rings.

3. Appendix: Two notions of weak nil-cleanness

Let R be an arbitrary associative ring with identity element 1 which differs from the zero element 0. The notations and a part of the terminology used in the current section are mainly in agreement with [1]. As above, $Id(R)$ denotes the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, and $J(R)$ the Jacobson radical of R.

Referring to the original source [3], a ring R is called *nil clean* if, for each $a \in R$, there are $q \in Nil(R)$ and $e \in Id(R)$ such that $a = q + e$ (see [10] for further information on this topic). In some literature, and especially in some recent important works (see, e.g., [11]), this concept is equivalently written by using the hyphen "$-$" like *nil-clean*.

This was substantially extended to the so-called weakly nil-clean rings in the commutative case [4] and in the general case [12] as follows: a ring R is said to be *weakly nil-clean* if, for every $a \in R$, there are $q \in Nil(R)$ and $e \in Id(R)$ such that $a = q + e$ or $a = q - e$.

Nevertheless, using the same notion in [13] were generalized both the classical π-regular rings and the defined above nil clean rings in the following way: a ring R is said to be *weakly nil clean* if, for any $a \in R$, there exist $e \in Id(R)$ and $q \in Nil(R)$ such that $a - e - q \in eRa$ (see, for more account, [10] as well). Note that this concept was originally written without the usage of the hyphen "$-$".

Resuming, both notions of weak nil-cleanness and weak nil cleanness expanded the notion of nil-cleanness (written as nil cleanness, too) in the sense of [3].

Reviewing the article [13], the reviewer in [14] was right to ask why the same notion is used as that in [12]. So, the objective of this section is to answer that question by using mathematical arguments only showing that one weak nil-cleanness is contained in the other weak nil cleanness as the evidences are not too obvious.

Our basic observation is the following one:

Proposition 4. *All weakly nil-clean rings in the sense of [12] are weakly nil clean in the sense of [13]. In other words, weakly nil-clean rings are always weakly nil clean.*

Proof. Utilizing the complete description of weakly nil-clean rings R, established independently and subsequently in [15] and [11], respectively, one writes that $R \cong R_1 \times R_2$, where R_1 is nil-clean and R_2 is a ring such that either $R_2 = \{0\}$ or $R_2/J(R_2) \cong \mathbb{Z}_3$ with nil $J(R_2)$. It can be checked that R_2 is strongly π-regular, itself. So, it follows directly from [13, Propositions 2.4 (ii), 3.2] that R is necessarily weakly nil clean, as claimed.

As a parallel verification of our initial assertion, the above decomposition for R implies that $R/J(R) \cong [R_1/J(R_1)] \times \mathbb{Z}_3$, where $R_1/J(R_1)$ is still nil-clean and, moreover, $J(R) \cong J(R_1) \times J(R_2)$ is nil because so is $J(R_1)$ (compare, resp., with [12] and [3]). Therefore, it follows immediately from [13, Propositions 2.4 (ii)] that $R/J(R)$ is weakly nil clean and hence so is R in view of [13, Proposition 2.8] since $J(R)$ is nil. □

As final comments, we may say that the next hopeful implications are fulfilled:

$$\text{nil clean} = \text{nil-clean} \Rightarrow \text{weakly nil-clean} \Rightarrow \text{weakly nil clean}.$$

4. Concluding Discussion and Open Questions

In conclusion, the next comments could be worthwhile.

Firstly, we ask of whether or not any semiprimitive (= Jacobson semi-simple) periodical ring (or even a semiprimitive (strongly) π-regular ring) is always von Neumann regular.

Secondly, we partially answer [16, Question 3.17] concerning those rings R such that for some fixed natural number $n \geqslant 2$ all elements of R satisfy the equation $x^n - x \in Nil(R)$. It is not too hard to verify that such a ring R is strongly π-regular. We, however, will detect a new property of these rings as follows: writing $(x^n - x)^m = 0$, we have $x^m(1 - x^{n-1})^m = 0$ and, consequently, $(x^{n-1})^m(1 - x^{n-1})^m = 0$, i.e., $(x^{n-1} - (x^{n-1})^2)^m = 0$. Then we can find an idempotent, say $e \in \mathbb{Z}[x]$ such that $x^{n-1} = e + t$, where $t \in Nil(R)$. Hence $[x(1-e)]^{n-1} = x^{n-1}(1-e) = t(1-e)$ is a nilpotent because t and e will commute as x and e do that. That is, $x - xe$ is a nilpotent. Q.E.D.

The next critical commentaries could be helpful to the interested in that subject reader.

Remark 1. It is worth to notice that [16, Theorem 2.6] is already well-known and is a simple consequence of [17, Theorem A1]. Indeed, $Nil(R)$ forms an ideal whence $Nil(R) = J(R)$ and thus the properties $P_n(R)$ and $Q_n(R)$ are equivalent at once, that is, $P_n(R) \iff Q_n(R)$. By the way, on line 6 of the Abstract in [16] there is a misprint, namely it should be “n is even with $n \not\equiv 1(mod\ 3)$” instead of “n is even with $n \equiv 1(mod\ 3)$”.

Finally, we would like to avoid some bugs by successfully correcting them in the next lines. Precisely, we correct the following issues:

Corrections. On p. 29, after Definition 1 from [5], there is a technical error, namely the element 0 has to be represented as $0 = 1 + (-1)$ with $1 = 1.1 \in 1P \cap P1$ (compare also with the truly given presentation of such an element, being an idempotent, stated at the end of p. 29). Also, on p. 30, line 3 in the proof of Proposition 1, the sign “$-$” is involuntarily omitted in the formulas which, however, does *not* affect the final conclusion. Moreover, on p. 31, line 4 of Remark 1 in [5] the intersection “$xR \cap eR = \{0\}$” must be “$xK \cap eK = \{0\}$”. And finally, on bottom of p. 32, “$x \in e + J(P)$” should be stated as “$x \in e + Nil(P)$”.

Likewise, on p. 709, at the beginning of line 3 of Proof in [6, Example 2.13] also there is a typo, namely “π regular” should be written as “π-regular”.

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Симметризация в чистых и ниль-чистых кольцах

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Мы вводим и исследуем *D-чистые* и *D-ниль-чистые кольца*, а также некоторые другие тесно связанные симметричные версии чистоты и ниль-чистоты. Дана исчерпывающая структурная характеристика для этих симметрично чистых и симметрично ниль-чистых колец в терминах радикала Джекобсона и его частного. Доказано, что сильно чистые (соответственно, сильно ниль-чистые) кольца всегда D-чистые (соответственно, D-ниль-чистые). Наши результаты подтверждают недавние публикации в Вестн. Иркутск. гос. ун-та, Матем. (2019) и Turk. J. Math. (2019). Мы также показываем, что слабо ниль-чистые кольца, определенные как в Danchev – McGovern (J. Algebra, 2015) и Breaz – Danchev – Zhou (J. Algebra & Appl., 2016), на самом деле слабо ниль-чистые в смысле Danchev – Šter (Taiwanese J. Math., 2015). Это отвечает на вопрос рецензента из-за D. Khurana (Math. Review, 2017).

Ключевые слова: L-чистые кольца, R-чистые кольца, D-чистые кольца, симметризация, слабо ниль-чистые кольца.

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